Small Ball Probabilities for the centered Poisson Process and Applications

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Abstract

For the centered jump Lévy processes we study the small ball probabilities. Namely, for the process X_ρ that is a centered pure jump Lévy process and that comes close to the Wiener process, when the intensity of jumps ρ tends to ∞, we are interested in the probability $\mathbf{P}\{\sup_{t\in[0,1]} |X_\rho(t)- f(t)| \leq r\}$ under $r \to 0$ for the admissible shift functions f. For this probability we obtained the asymptotic estimate that coincides with the known Wiener-type estimate, when the intensity ρ increases "fast". This asymptotic estimate is different, when the intensity changes "slower".

We discuss also various applications of such type estimates, for example, to the proof of the functional Strassen law of the iterated logarithm for the empirical processes, to the estimation of the quantization error in the approximation theory, the coding theory and to the theory of reliability.

1 Statement of the problem

Let $(\mathrm{B}[0,1],\ \|\cdot\|)$ be the space of bounded functions on $[0,1]$ endowed with the uniform norm and let U be the unit ball in this space. Let us consider the space of admissible shifts of the Wiener measure on [0,1]. It is a Hilbert space $(\mathrm{E}, |\cdot|_{\mathrm{E}})$ defined as

$$\mathrm{E} := \{f \in \mathrm{B}[0,1] : f(0) = 0,\ f \in \mathrm{AC}[0,1],\ |f|_{\mathrm{E}} < \infty\},$$

where $\mathrm{AC}[0,1]$ denotes the space of absolutely continuous functions on $[0,1]$ and $|\cdot|_{\mathrm{E}}$ is a Hilbert norm

$$|f|^2_{\mathrm{E}} := \int_0^1 f'(t)^2\, dt.$$

Let us introduce a jump Lévy process ξ associated to the measure of jumps $\mathcal{P}$, where $\mathcal{P}$ is a Poisson random measure on the space $\mathbf{R}^+ \times \mathbf{R} \setminus \{0\}$ defined by a Lévy measure $dt \times \Lambda(d\ell)$, i.e.,

$$\xi(t) = \int_0^t \int_{\mathcal{L}} \ell\, \mathcal{P}(dt, d\ell). \tag{1}$$

We assume that the deterministic measure $\Lambda(d\ell)$ is concentrated on a bounded subset $\mathcal{L} \subset \mathbf{R} \setminus \{0\}$. The expectation and the variance of the process ξ are determined by the expressions

$$\mathbf{E}\xi(t) = t \int_{\mathcal{L}} \ell\, \Lambda(d\ell), \quad \mathbf{D}\xi(t) = t \int_{\mathcal{L}} \ell^2\, \Lambda(d\ell).$$

We assume that $\mathbf{E}\xi(1)$ is finite and denote $\sigma^2 := \mathbf{D}\xi(1)$.

Now by using the process ξ we construct the centered normalized process of intensity $\rho > 0$ as follows

$$\mathrm{X}_\rho(t) := \rho^{-1/2}(\xi(\rho t) - \rho\mathbf{E}\xi(t)), \quad t \in [0,1].$$

For the process X_ρ the invariance principle holds, i.e., $X_\rho \overset{d}{\Rightarrow} \sigma W$, when $\rho \to \infty$.

In this work we investigate the limit behavior of the shifted small ball probabilities of the process X_ρ, i.e., we study

$$\mathbf{P}\{X_\rho - \lambda f \in r\mathrm{U}\} \text{ under } \rho \to \infty,\ \lambda \to \infty,\ r \to 0.$$

We consider only admissible shift functions f. The space of admissible shifts of X_ρ is known to be a subset of E, the space of admissible shifts of the Wiener measure.

The previous result in this direction:

K.L.Chung (1964), unshifted small ball probability ($f \equiv 0$) for the Wiener process W

$$\mathbf{P}\{\mathrm{W} \in r\mathrm{U}\} = \exp\left\{-\frac{\pi^2}{8r^2}(1+o(1))\right\}, \quad \text{when } r \to 0.$$

K.Grill (1991), shifted small ball probability for W

$$\mathbf{P}\{\mathrm{W} - \lambda f \in r\mathrm{U}\} = \exp\left\{-\frac{\lambda^2}{2}\,|f|^2_{\mathrm{E}} - \frac{\pi^2}{8r^2}(1+o(1)) + \mathbf{R}\right\}, \quad \text{when } \lambda \to \infty,\ r \to 0, \tag{2}$$

where the term $\mathbf{R} = \mathbf{R}(r, \lambda, f) = o(\lambda^2)$ can be bigger than r^{-2} under some relations between r and λ

For the theory of the small ball probabilities of the gaussian processes see Li, Shao (2002).

A.A.Mogulskii (1974), unshifted small ball probability for X_ρ

$$\mathbf{P}\{X_\rho \in r\mathrm{U}\} = \exp\left\{-\frac{\pi^2\sigma^2}{8r^2}(1+o(1))\right\}, \text{ when } \rho r^2 \to \infty,\ r \to 0.$$

The results presented below generalizes the Mogulskii formula.

Since the invariance principle holds, it is natural to expect that the behavior of the small ball probabilities for the process X_ρ is similar to the behavior of this probability for the Wiener process. However, Theorem 1 below shows that it is not always true.

2 Result

The proof of Theorem 1 is based on the properties of the pure jump Lévy processes. We did not use the invariance principle, i.e., KMT inequality (Komlós, Major, Tusnády (1975)), that would lead the proof (under the strong restriction $\rho^{1/2} r \lambda^{-2} \to \infty$) to the formula (2) for the Wiener process. We use a new approach based on the Skorokhod formula for the Poisson measures (Skorokhod (1964)).

Theorem 1 *Let $f \in \mathrm{E}$ such that $\mathrm{Var} f' < \infty$. If three conditions hold*

1. $\lambda \to \infty$, $\rho \to \infty$, $r/\lambda \to 0$;
2. $\rho\, r^2 \to \infty$;
3. $\rho/\lambda^2 \to \infty$

and $\delta \in [0, 1)$ is arbitrary, then the following estimates are true

$$\mathbf{P}\{X_\rho - \lambda f \in r\mathrm{U}\} \le \exp\left\{-\frac{\lambda^2}{2\sigma^2}\,|f|^2_{\mathrm{E}} - \frac{\pi^2\sigma^2}{8r^2}\,(1+o(1)) + \frac{\lambda r}{\sigma^2}\,(f'(1) + \mathrm{Var} f')(1+o(1)) + \frac{1}{6\sigma^6}\int_{\mathcal{L}} \ell^3 \Lambda(d\ell) \cdot \frac{\lambda^3}{\rho^{1/2}}\left(\int_0^1 f'^3(t)\,dt + o(1)\right)\right\},$$

$$\mathbf{P}\{X_\rho - \lambda f \in r\mathrm{U}\} \ge \exp\left\{-\frac{\lambda^2}{2\sigma^2}\,|f|^2_{\mathrm{E}} - \frac{\pi^2\sigma^2}{8r^2(1-\delta)^2}(1+o(1)) + (2\delta - 1)\frac{\lambda r}{\sigma^2}(f'(1) + \mathrm{Var} f')(1+o(1)) + \frac{1}{6\sigma^6}\int_{\mathcal{L}} \ell^3 \Lambda(d\ell) \cdot \frac{\lambda^3}{\rho^{1/2}}\left(\int_0^1 f'^3(t)\,dt + o(1)\right)\right\}.$$

Comments:
1. The third term of the estimate obtained in the theorem is the same that the term $\mathbf{R}$ in the Grill formula (2). The last term $\lambda^3\rho^{-1/2}\int_{\mathcal{L}}\int_0^1 f'^3(t)\,\ell^3\Lambda(d\ell)dt$ is new. It appears because we consider jump (Poisson) processes. Note, the asymptotics for X_ρ are different from the Wiener-type asymptotics (2), when the new term is bigger than the second and third terms, i.e., when $\rho \ll \lambda^6\, r^4$ and $\rho \ll \lambda^4\, r^{-2}$. So, if we assume

$$\rho = \lambda^p,\ p > 2, \qquad r = \lambda^{-z},\ z > -1,$$

then we illustrate the result of the theorem as follows:

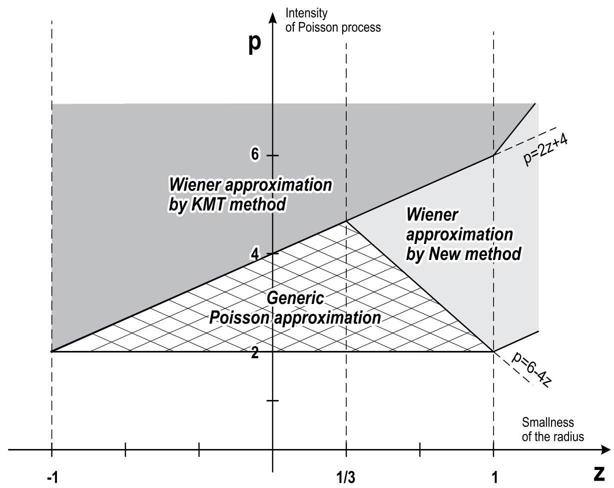

Figure 1: The domain of the generic asymptotics for the jump (Poisson) processes.

This effect is the sequence of that the process X_ρ becomes sufficiently different from W, when the intensity ρ increases slowly.
2. Notice, if we consider the result of the theorem for a jump Lévy process with a symmetrical jump measure Λ, for example, $\Lambda = \delta_{-m}\bigcup\delta_m,\ m \in \mathbf{R}^+$, then the generic "Poisson" term of the estimate disappears. Hence, we infer that non-symmetry of the sample paths of the centered process X_ρ is the reason of the difference from the Wiener case.

3 Applications

1. The rates of convergence in the functional Strassen law of the iterated logarithm are related to shifted small ball probabilities. Making use of Theorem 1 we obtained the rates in the analogue of the Strassen law (see Mason (1986), Berthet(1996), Deheuvels(2000)) for the tail empirical process α_n^* ($\alpha_n^*(t) = n^{1/2}h_n^{-1/2}\,\alpha_n(h_n t),\ t \in [0,1]$, where α_n is the standard empirical process, $h_n \downarrow 0, n \to \infty$), i.e., we found ℓ and C in the expression

$$\liminf_{n\to\infty}\ (\log\log n)^{\ell}\,\|\alpha_n^*(\cdot)/(2\log\log n)^{1/2} - f(\cdot)\| = C.$$

This result can be illustrated by the figure 2. See for details Shmileva (2004).
2. Let P be the distribution of X_ρ on E and N be a positive integer. It is interesting to find an optimal approximation of P by a probability measure which is supported by N points $e_1, ...e_N \in \mathrm{E}$. The quantization error in this case is the following:

$$\delta(\rho, N, q) := \inf\{\mathbf{E}\min_{i=1..N}\|\mathrm{X}_\rho - e_i\|^q | e_1, ...e_N \in \mathrm{E}\}$$

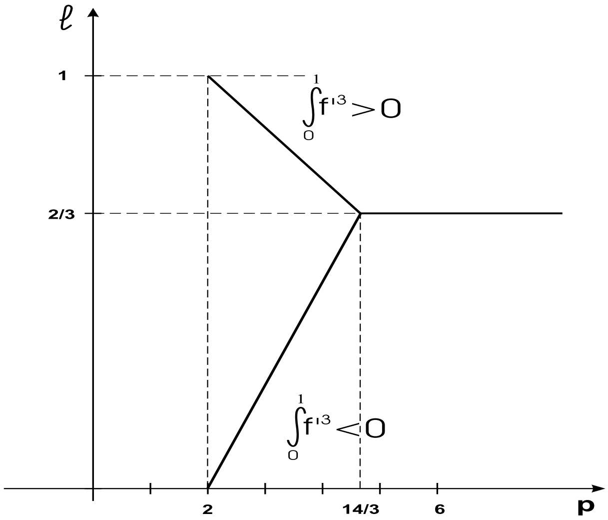

Figure 2: ℓ : $(\log\log n)^{-\ell}$ —the rate of convergence, p : $h_n = n^{-1}(\log\log n)^{p/2}$ — the sequence, which define the process α_n^*. The rate of convergence is the same as for W, when h_n decreases "slow". When h_n changes "faster", the rate is bigger or smaller in dependence of the sign of $\int_0^1 f'^3$.

If we consider N independent random elements $Y_1, ...Y_N$ with the distribution P, then it is reasonable to consider the following value

$$\Delta(\rho, N, q) := (\mathbf{E}\,\mathbf{E}\{\min_{i=1..N} \|\mathrm{X}_\rho - Y_i\|^q | Y_1, Y_2 \dots Y_N\})^{1/q}.$$

It is closely connected to shifted small ball probabilities

$$\Delta(\rho, N, q)^q = \int_0^\infty \int_{\mathrm{E}} (1 - \mathbf{P}\{\mathrm{X}_\rho - x \in \varepsilon^{1/q}\,\mathrm{U}\})^N P(dx)\, d\varepsilon.$$

For the result see Dereich, Lifshits (2004). For the others applications of small ball probabilities see Li, Shao (2002).

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