Simple Repairable Continuous State Systems of Continuous State Components

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Abstract

The output of the component's performance is characterized by a monotone deterministic function or by a stochastic process with monotone paths. Stationary expected output and the probability of exceeding the given level of an output are derived for perfectly repaired components. Simple systems of components with the continuous output are considered. The case of imperfect repair is discussed.

1. Introduction

The output of binary components and systems is described by 0 when they are "off" and 1, when they are "on". Reliability theory of engineering systems is mostly devoted to this case, although the multistate components and systems were also considered rather extensively (see Lisniansky and Levitin (2003) for references). Less attention was paid to the continuous output case, as its modeling is not a straightforward generalization of the multistate one.

Let the performance of a component be characterized by some positive valued decreasing (increasing) function of performance $Q(t)$ ($\Phi(t)$) or by the corresponding stochastic process $Q_t, t \geq 0$ ($\Phi_t, t \geq 0$), showing monotonic deterioration in time of the component's output. The monotonicity assumption is quite natural in many applications. We shall consider the following settings, which will be used in Section 2 for modeling the output of a repairable component:

a. Let a system consist of two independent components in series. The first component is binary with Cdf $F(t)$ and the second is a continuous state component which state is defined by the decreasing $Q(t)(Q_t, t \geq 0)$. It is clear that expectation of the output of this system is given by

$$Q_E(t) = \overline{F}(t)Q(t)(Q_E(t) = \overline{F}(t)E[Q_t]), \tag{1}$$

where we assume that $E[Q_t]$ exists and that the criterion of failure with respect to the level of $Q(t)(Q_t, t \geq 0)$ is not defined (as usually: $\overline{F}(t) \equiv 1 - F(t)$).

b. There is only one component with increasing output $\Phi_t, t \geq 0$ and the failure is defined as exceeding some deterministic level φ_0. Consider a stochastic process $\xi_t, t \geq 0$, which is equal to zero when initial process exceeds φ_0:

$$\xi_t = \Phi_t I(\varphi_0 - \Phi_t), \tag{2}$$

where $I(t)$ is the corresponding indicator. Therefore, the expectation of an output at time t is $\Phi_E(t) = E[\xi_t]$. Without loosing generality, assume that $\Phi_0 = \Phi(0) = 0$, as in the case of accumulated damage (degradation).

c. The same as in b. but the critical level φ_0 is random variable Ψ_0 (with the Cdf $F_0(\varphi)$) independent of $\Phi(t)$.

2. Asymptotic performance of repairable components

a. Let each failure of the component be instantaneously perfectly repaired at failures. This means that all inter-failure times are i.i.d. with distribution functions $F(t)$, whereas the output after each failure is ideally 'refreshed'. Denote by $\widetilde{Q}(t)$ the random value of the output at time t. Therefore:

$$E[\widetilde{Q}(t)] \equiv Q_E(t) = \overline{F}(t)E[Q_t] + \int_0^t h(x)\overline{F}(t-x)]E[Q_{t-x}]dx, \tag{3}$$

where $h(x)$ is the renewal density function for the ordinary renewal process governed by the distribution function $F(t)$. Applying the key renewal theorem to equation (3), the stationary value (as $t \to \infty$) of the expected output is obtained in a usual way:

$$Q_{ES} = \frac{1}{\overline{T}}\int_0^\infty \overline{F}(x)E[Q_x]dx, \tag{4}$$

where $\overline{T}$ is the mean time to failure.

The probability that the output exceeds some acceptable level $q_0 > 0$ is also of interest. Similar to (3) and (4), the following stationary probability can be obtained:

$$P_S(q_0) = \frac{1}{\overline{T}}\int_0^\infty \overline{F}(x)\Pr(Q_x \geq q_0)dx = \frac{1}{\overline{T}}\int_0^\infty \overline{F}(x)(1 - F(x, q_0))dx \tag{5}$$

where $F(x, q_0)$ is the distribution of the first passage time for the decreasing stochastic process $Q_t, t \geq 0$.

b. Denote by $\theta_t(\varphi)$ the pdf of Φ_t for the fixed t. It is clear that it can be easily defined for processes for which the distribution of the first passage time can be obtained. Then:

$$E[\xi_t] = \int_0^{\varphi_0} \varphi\theta_t(\varphi)d\varphi. \tag{6}$$

Assume that this mean exists for $\forall t \geq 0$. The perfect repair is instantaneously performed each time Φ_t reaches φ_0. Therefore, the inter-arrival times of the renewal process are just the corresponding i.i.d. first passage times with the Cdf $F(t, \varphi_0)$ and the mean $\overline{T}_{\varphi_0}$. It can be shown that $E[\xi_t]$ is decreasing in t and that its integral from 0 to infinity is finite. Therefore the key renewal theorem can be applied:

$$\Phi_{ES}(\varphi_0) = \frac{1}{\overline{T}_{\varphi_0}}\int_0^\infty E[\xi_t]dt \tag{7}$$

The similar result can be obtained for the stationary probability for the output values to be in the interval $[\varphi', \varphi_0]$ for some $0 < \varphi' < \varphi_0$.

c. Let now φ_0 be random with the Cdf $F_0(\varphi)$ and consider firstly deterministic $\Phi(t)$. As this function is increasing, the inter-arrival times are defined by the following distribution

$$P(\varphi_0 \leq \Phi(t)) = F_0(\Phi(t)) \tag{8}$$

and, eventually:

$$\Phi_{ES} = \frac{1}{\overline{T}_0} \int_0^\infty \overline{F}_0(\Phi(x))\Phi(x)dx. \tag{9}$$

For the random process equation (9) is modified to:

$$\Phi_{ES} = \frac{1}{\overline{T}_0} \int_0^\infty E[\overline{F}_0(\Phi_x)]E[\Phi_x]dx, \tag{10}$$

where $E[\overline{F}_0(\Phi(t))]$ is the inter-arrival time distribution and $\overline{T}_0$ is its mean.

3. Simple systems

The above reasoning can be generalized to systems of components of the described type. We shall consider the setting "a." and stationary characteristics but the following is valid for other settings and for $\forall t \geq 0$ as well. Let $h(x_1, x_2, ..., x_n)$ be an "ordinary' structure function of a coherent system of n binary components. Assume now that each component of this system is not binary any more, but a repairable continuous state one with an output defined by relations (4) and (5). Let the following rule be applied to the output of our system (Barlow and Wu 1978):

$$\widetilde{Q}(t) = \max_{r=1,2,...,l,;i\in\Omega_r} \min \widetilde{Q}_i(t), \tag{11}$$

where $\widetilde{Q}_i(t)$, is an output of the i th component $i = 1,2,...,n$ and $\Omega_1, \Omega_2, ..., \Omega_l$ are the corresponding minimal path sets. It is clear that the parallel continuous output system is defined by: $\widetilde{Q}(t) = \max_{1\leq i\leq n} \widetilde{Q}_i(t)$, whereas relation $\widetilde{Q}(t) = \min_{1\leq i\leq n} \widetilde{Q}_i(t)$ defines the series system. Denote by $P_i(q_0, t) \equiv P(\widetilde{Q}_i(t) \geq q_0)$. It is clear that, if relation (13) holds, then

$$P(\widetilde{Q}(t) \geq q_0) \equiv P(q_0, t) = h(P_1(q_0, t), P_2(q_0, t), ..., P_n(q_0, t)) \tag{12}$$

for $\forall t \geq 0$ and, specifically, for the stationary value when $t \to \infty$. The expected system output in this case is:

$$E[\widetilde{Q}(t)] = \int_0^\infty P(q_0, t)dq_0. \tag{13}$$

4. Imperfect repair

The results of previous sections were derived under the usual assumption that repair is ideal, thus returning the system to 'as good as new' state. We shall consider here only one type of imperfect repair and apply it to Model "c". Another type of imperfect repair, which is defined for binary components as the combination of the perfect repair and the minimal repair (Block *et all* 1985), can be easily generalized to the case of the continuous output in Model "a".

Assume that after each repair the process $\Phi_t, t \ge 0$ is restarted from the same initial level, but the critical level of the output, which is a random variable and defines a failure (end of a cycle), is stochastically decreasing with each repair. This kind of imperfect repair is relevant in various applications. Specifically, let $\Psi_{0,i}, i = 1,2,...$ be the critical output at the end of the ith cycle. For the perfect repair all $\Psi_{0,i}, i = 1,2,...\Psi_{0,1} \equiv \Psi_0$ are i.i.d. Let the sequence $\Psi_{0,i}, i = 1,2,...$ be stochastically decreasing:

$$\Psi_{0,i+1} \le_{st} \Psi_{0,i} \Leftrightarrow \overline{F}_{0,i+1}(\varphi) \le \overline{F}_{0.i}(\varphi),\ i = 1,2,...,\ \forall \varphi \in [0,\infty), \tag{14}$$

where, $F_{0,i}(\varphi)$ is the Cdf of $\Psi_{0,i}$ and $F_{0,1}(\varphi) \equiv F_0(\varphi)$.

Consider the following specific model. The duration of the first cycle is defined by the Cdf $F_{0,1}(\varphi) \equiv F_0(\varphi)$, whereas $\Psi_{0,1} \equiv \Psi_0$. Let the next cycles be governed by the distribution $F_{0,i}(\varphi) = E[F_0(\varphi \mid \tilde{\varphi}_i)]$ where $\tilde{\varphi}_i$ is a random starting value, and

$$\overline{F}_0(\varphi \mid \tilde{\varphi}_i) = \frac{\overline{F}_0(\varphi + \tilde{\varphi}_i)}{\overline{F}_0(\tilde{\varphi}_i)}$$

Assume, specifically, that the starting age of this conditional distribution is the decreased output at the end of the previous cycle.

$$F_{0,i+1}(\varphi) = E[F_0(\varphi \mid q\Psi_{0,i})], i = 1,2,... \tag{15}$$

where $0 \le q < 1$ shows 'the extent' of the repair action. The value $q = 0$ corresponds to the perfect repair and it is clear that there is no analogue of the minimal repair (Finkelstein 1992) in this case. Therefore, this setting is similar to the Model 1 of Kijima (1989). It can be proved (Finkelstein 2000), that if $F_0(\varphi)$ is IFR, then (14) holds, and the limiting distribution $\lim_{i\to\infty} F_{0,i}(\varphi) = F_l(\varphi)$ exists (for the latter result the IFR assumption is not required). Therefore, relation (10) for the stationary output value (substituting $F_0(\varphi)$ by $F_l(\varphi)$) holds.

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