Sensitivity Analysis in Reliability Based Optimization

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Abstract

Reliability based optimization design consists of selecting the values of the design variables that minimize a cost function subject to some reliability conditions, and geometric and code constraints. This involves the definition of the joint probability density of the random variables involved by means of statistical parameters (means, standard deviations, correlation coefficients, covariance matrices, etc.) that are considered as data. This paper shows how a sensitivity analysis can be performed, i. e., how to determine the rate of change of the cost and reliability indices due to data changes. To this end data values are converted into fixed artificial variables and then, the sensitivities become the values of the dual variables associated with them in the original optimization problem. The method is illustrated by its application to a practical case.

1 Introduction

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be the set of variables involved in the design and reliability analysis of an engineering work. These variables include geometric variables, material properties, loads, standard deviations, lower safety factor bounds, lower reliability index bounds, etc. By application of first-order reliability methods (FORM) (Madsen, Krenk, and Lind 1986), and optimization techniques (Castillo, Mínguez, Ruíz-Terán, and Fernández-Canteli 2003), (Castillo, Losada, Mínguez, Castillo, and Baquerizo 2004) and (Castillo, Mínguez, Losada, and Castillo 2003), it is possible to formulate reliability-based optimization models that can be written as

$$\underset{\tilde{\boldsymbol{x}}}{\text{Minimize}} \quad c(\tilde{\boldsymbol{x}}) \qquad \text{subject to} \qquad \left\{ \begin{array}{rcl} g_i(\tilde{\boldsymbol{x}}) & \geq & 0;\ \forall i \in I, \\ h_i(\tilde{\boldsymbol{x}}) & \geq & 0;\ \forall i \in I, \\ r_j(\tilde{\boldsymbol{x}}) & \leq & 0;\ \forall j \in J, \end{array} \right. \tag{1}$$

where $\tilde{\boldsymbol{x}}$ denotes the mean, characteristic, or fixed values of the variables, $c(\tilde{\boldsymbol{x}})$ is the objective function to be optimized (cost function), $g_i(\tilde{\boldsymbol{x}}) \geq 0$ are the limit state equations related to the different failure modes including safety factor lower bounds (F_i^0), $h_i(\tilde{\boldsymbol{x}}) = \beta_i(\tilde{\boldsymbol{x}}) - \beta_i^0 \geq 0$ are constraints that fix the lower bounds (β_i^0) on the reliability indices, and $r_j(\tilde{\boldsymbol{x}}) \leq 0$ are the geometric or code constraints including bounds on variables. Note that we make no distinction between random and deterministic variables; all variables involved are assumed random because deterministic variables are only particular cases of random variables.

In what follows, $\boldsymbol{A}$ denotes the vector that contains the fixed variables in $\boldsymbol{X}$ (non-optimization variables). The elements of $\boldsymbol{A}$ are viewed as parameters. The remaining variables in $\boldsymbol{X}$ will be the optimization variables. After obtaining the optimal solution $(\tilde{\boldsymbol{x}}^*, \tilde{\boldsymbol{a}})$ of problem (1), which provides the optimal cost $c(\tilde{\boldsymbol{x}}^*, \tilde{\boldsymbol{a}})$ fulfilling some reliability requirements, we need to evaluate the quality of the reliability model. Part of this quality evaluation includes sensitivity analysis. Sensitivity analysis is the study of how variations in the output of a model can be apportioned, qualitatively or quantitatively, to different sources of variation. As a whole, sensitivity analysis is used to increase the confidence in the model and its predictions, by providing an understanding of how the model response variables are affected by changes in the inputs. Adding a sensitivity analysis to a study means adding quality to it.

2 Optimal solution of reliability based optimization problems

Before performing the sensitivity analysis of problem (1) we have to solve it and get the optimal solution $(\tilde{\boldsymbol{x}}^*, \tilde{\boldsymbol{a}})$. Unfortunately, that problem cannot be solved directly because each of the constraints $h_i(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}}) =$

$\beta_i(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}}) - \beta_i^0 \geq 0;\ \forall i \in I$ involves another optimization problem, i.e.,

$$\beta_i(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}}) = \underset{\boldsymbol{x}_i, \boldsymbol{a}_i}{\text{Minimum}} \quad \sqrt{\sum_{j=1}^{n} z_{ij}^2} \qquad \text{subject to} \qquad \left\{ \begin{array}{rcl} g_i^*(\boldsymbol{x}_i, \boldsymbol{a}_i) & = & 0, \\ T(\boldsymbol{x}_i, \boldsymbol{a}_i, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}}) & = & \mathbf{z}_i, \end{array} \right. \tag{2}$$

where $(\boldsymbol{x}_i, \boldsymbol{a}_i)$ are the design random variables for failure mode i, $g_i^*(\boldsymbol{x}_i, \boldsymbol{a}_i) = 0$ is the limit state equation for mode i defining strict failure ($F_i^0 = 1$), and $T(\boldsymbol{x}_i, \boldsymbol{a}_i, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}})$ is the usual transformation (Rosenblatt 1952), (Nataf 1962) that converts $\boldsymbol{x}_i$ and $\boldsymbol{a}_i$ into the standard independent normal random variables $\mathbf{z}_i$.

An alternative formulation for solving this problem iteratively is proposed by (Castillo, Conejo, Mínguez, and Castillo 2003):

Master Problem: For iteration ν

$$\underset{\tilde{\boldsymbol{x}}}{\text{Minimize}} \quad c(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}}) \qquad \text{subject to} \qquad \left\{ \begin{array}{rcl} g_i(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}}) & \geq & 0;\ \ \forall i \in I, \\ \beta_i^{(s)} + \boldsymbol{\lambda}_i^{(s)^T} (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{x}}^{(s)}) & \geq & \beta_i^0;\ \ \forall i \in I;\ \ s = 1, \cdots, \nu - 1, \\ r_j(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}}) & \leq & 0;\ \forall j \in J, \end{array} \right. \tag{3}$$

where we obtain $\tilde{\boldsymbol{x}}^{(\nu)}$.

Subproblem: $\forall i$

$$\beta_i^{(\nu)} = \underset{\boldsymbol{x}_i, \boldsymbol{a}_i, \tilde{\boldsymbol{x}}}{\text{Minimum}} \quad \sqrt{\sum_{j=1}^{n} z_{ij}^2} \qquad \text{subject to} \qquad \left\{ \begin{array}{rcl} g_i^*(\boldsymbol{x}_i, \boldsymbol{a}_i) & = & 0, \\ T(\boldsymbol{x}_i, \boldsymbol{a}_i, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}}) & = & \mathbf{z}_i, \\ \tilde{\boldsymbol{x}} & = & \tilde{\boldsymbol{x}}^{(\nu)} :\ \ \boldsymbol{\lambda}_i^{(\nu)}. \end{array} \right. \tag{4}$$

The process of solving iteratively these two problems is repeated, starting from $\nu = 0$ and increasing the value of ν by one, until convergence. Note that at iteration $\nu = 0$ there is no hyperplane approximation of constraint $h_i(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}}) \geq 0;\ \forall i \in I$. It should be noted that problem (3) is a relaxation of problem (1) in the sense that functions $h_i(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{a}}) \geq 0$ are approximated using cutting hyperplanes, which implies that problem (3) reproduces problem (1). Observe also that cutting hyperplanes are constructed using the dual variable vector ($\boldsymbol{\lambda}_i^{(\nu)}$) associated with constraints $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{x}}^{(\nu)}$ (the subproblems).

3 Sensitivity analysis

The problem of sensitivity analysis in reliability based optimization has been discussed by several authors, see, for example, (Enevoldsen 1994), or (Sorensen and Enevoldsen 1992). In this section we show how the duality methods can be applied to sensitivity analysis in a straightforward manner. We emphasize here that the method to be presented in this section is of general validity. The basic idea is simple. Assume that we wish to know the sensitivity of the objective function to changes in some data values. Converting the data into artificial variables and locking them, by means of constraints, to their actual values, we obtain a problem that is equivalent to the initial optimization problem but has a constraint such that the values of the dual variables associated with them give the desired sensitivities.

Consider the following general nonlinear *primal problem* (P):

$$\underset{\boldsymbol{x}}{\text{Minimize}} \quad Q_P = f(\boldsymbol{x}, \boldsymbol{a}) \qquad \text{subject to} \qquad \left\{ \begin{array}{rcl} \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) & = & \boldsymbol{b}, \\ \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) & \leq & \boldsymbol{c}, \end{array} \right. \tag{5}$$

where $\boldsymbol{x} \in \mathbb{R}^t$, $\boldsymbol{a} \in \mathbb{R}^t$, $\boldsymbol{b} \in \mathbb{R}^p$, $\boldsymbol{c} \in \mathbb{R}^q$. Every primal nonlinear programming problem P, which is stated as in (5), has an associated dual problem D, which is defined as:

$$\underset{\boldsymbol{\lambda}, \boldsymbol{\mu};\ \boldsymbol{\mu} \geq \boldsymbol{0}}{\text{Maximize}} \ Q_D = \ \text{Inf}_{\boldsymbol{x}} \ \{\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})\} \tag{6}$$

where

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = f(\boldsymbol{x}, \boldsymbol{a}) + \boldsymbol{\lambda}^T \left(\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) - \boldsymbol{b}\right) + \boldsymbol{\mu}^T \left(\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) - \boldsymbol{c}\right), \tag{7}$$

is known as the Lagrangian function associated with the primal problem (5), and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are called dual variables and they are vectors of dimensions p and q, respectively. Note that only the dual variables ($\boldsymbol{\mu}$ in this case) associated with the inequality constraints ($\boldsymbol{g}(\boldsymbol{x},\boldsymbol{a})-\boldsymbol{c}$, in this case), must be nonnegative.

Theorem 1 (Objective function sensitivities with respect to the parameter a) *The sensitivity of the objective function of the primal problem (5) with respect to the parameter* $\mathbf{a}$ *is given by*

$$\nabla_{\boldsymbol{a}}\mathcal{L}(\boldsymbol{x}^*,\boldsymbol{\lambda}^*,\boldsymbol{\mu}^*,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}), \tag{8}$$

which is the partial derivative of its Lagrangian function (7) with respect to $\boldsymbol{a}$ *evaluated at the optimal solution* $\boldsymbol{x}^*,\boldsymbol{\lambda}^*$*, and* $\boldsymbol{\mu}^*$*.*

To apply the general method of sensitivity analysis to reliability based optimization problems (1), problems (3) and (4) are transformed into the following ones using the auxiliary variables $\boldsymbol{y}$

Master Problem: For last iteration ν^*

$$\underset{\tilde{\boldsymbol{x}},\boldsymbol{y}}{\text{Minimize}} \quad c(\tilde{\boldsymbol{x}},\boldsymbol{y}) \qquad \text{subject to} \qquad \left\{\begin{array}{rcl} g_i(\tilde{\boldsymbol{x}},\boldsymbol{y}) & \geq & 0;\ \forall i\in I, \\ \beta_i^{(\nu^*)}+\boldsymbol{\lambda}_i^{(\nu^*)^T}(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{x}}^{(\nu^*)})+\boldsymbol{\eta}_i^{(\nu^*)^T}(\boldsymbol{y}-\boldsymbol{y}^{(\nu^*)}) & \geq & y_i^0;\ \forall i\in I, \\ r_j(\tilde{\boldsymbol{x}},\boldsymbol{y}) & \leq & 0;\ \forall j\in J, \\ \boldsymbol{y} & = & \tilde{\boldsymbol{a}}:\ \boldsymbol{\mu}, \end{array}\right. \tag{9}$$

where y_i^0 corresponds to β_i^0, and

Subproblem: $\forall i$

$$\beta_i^{(\nu^*)} = \underset{\boldsymbol{x}_i,\boldsymbol{a}_i,\tilde{\boldsymbol{x}},\boldsymbol{y}}{\text{Minimum}} \sqrt{\sum_{j=1}^{n} z_{ij}^2} \qquad \text{subject to} \qquad \left\{\begin{array}{rcl} g_i^*(\boldsymbol{x}_i,\boldsymbol{a}_i) & = & 0 \\ T(\boldsymbol{x}_i,\boldsymbol{a}_i,\tilde{\boldsymbol{x}},\boldsymbol{y}) & = & \mathbf{z}_i \\ \tilde{\boldsymbol{x}} & = & \tilde{\boldsymbol{x}}^{(\nu^*)}:\ \boldsymbol{\lambda}_i^{(\nu^*)} \\ \boldsymbol{y} & = & \tilde{\boldsymbol{a}}:\ \boldsymbol{\eta}_i^{(\nu^*)} \end{array}\right. \tag{10}$$

where dual variables $\boldsymbol{\mu}$ are the sensitivities of the objective function value to changes in the parameters $\tilde{\boldsymbol{a}}$, the dual variables $\boldsymbol{\eta}_i^{(\nu^*)}$ give the sensitivities of the reliability indices β_i to $\tilde{\boldsymbol{a}}$, and $\boldsymbol{\lambda}_i^{(\nu^*)}$ give the sensitivities of the reliability indices β_i with respect to the optimal values of variables $\tilde{\boldsymbol{x}}$. In other words, the values of the dual variables give the rate of change of the corresponding objectives functions due to small increments of the corresponding data parameter.

Remark 1 *Note that problems (9) and (10) need to be solved only once, i.e., after solving problem (3) and (4). Because the starting point is already the optimal solution, convergence is ensured at the first iteration.*

4 Example

Consider a tubular steel column of $h=25\ m$ height loaded by a vertical load P. Its cross sectional area has diameter d and thickness t. The column is optimized by minimizing the direct cost $c_d=c_y h\pi dt$ using d and t as optimization variables, where $c_y=20000\ \$/m^3$ is the steel cost.

The reliability requirements are related to the limit state functions for material yielding and global (Euler) buckling. The lower reliability indices and safety factors are chosen as $\beta_y^0=4$, $\beta_b^0=4$ and $F_y^0=1.5$, $F_b^0=1.5$, respectively.

$$\textit{Material yielding:} \quad g_y=\frac{f_y\pi dt}{P}-F_y^0, \qquad \textit{Global buckling:} \quad \left\{\begin{array}{c} g_b=\dfrac{(\gamma-\sqrt{\gamma^2-1/\lambda_e^2})f_y\pi dt}{P}-F_b^0, \\ \gamma=(\lambda_e^2+k_i(\lambda_e-0.2)+0.8)/(2\lambda_e^2), \\ \lambda_e=\dfrac{h}{0.35d\pi}\sqrt{\dfrac{f_y}{E}}, \end{array}\right.$$

where f_y is the yield stress, E is the Young modulus and γ and λ_e are auxiliary parameters for modelling the global buckling for a cold-formed, welded and non-stress relieved steel cylinder, taking into account the defects with the imperfection parameter k_i.

It is assumed that the uncertainties are connected with the load P, the yield stress f_y, the Young modulus E and the imperfection parameter k_i, which are modelled as stochastic variables as shown in the table below.

Variable	Meaning	Distribution	Mean	Coefficient of variation
P	Vertical load	N	$10\ MN$	20 %
E	Young's modulus	LN	$2.1 \cdot 10^5\ MN/m^2$	6 %
f_y	Yield stress	LN	$650\ MN/m^2$	5 %
k_i	Imperfection parameter	N	0.49	10 %

The optimal solution of this problem is $c_d^* = 14460.16$ \$, $d^* = 2.08\ m$ and $t^* = 0.0044\ m$. With optimal reliability indices $\beta_y^* = 4$, $\beta_b^* = 4$, equal to the lower bounds.

Performing a complete sensitivity analysis as in Section 3, the following results are obtained:

	∂c_d^*	β_y^*	β_b^*		∂c_d^*	β_y^*	β_b^*
∂d	$--$	3.816	5.722	∂v_P	30606.479	-16.839	-16.859
∂t	$--$	1801.935	1803.477	∂v_E	0.000	0.000	-1.060
$\partial \mu_P$	1446.017	-0.796	-0.796	∂v_{f_y}	23672.180	-13.024	-7.430
$\partial \mu_E$	0.000	0.000	0.000	∂v_{k_i}	0.000	0.000	-2.145
$\partial \mu_{f_y}$	-22.246	0.012	0.009	∂F_y^0	0.000	$--$	$--$
$\partial \mu_{k_i}$	0.000	0.000	-5.164	∂F_b^0	0.000	$--$	$--$
∂c_i	0.723	0.000	0.000	$\partial \beta_y^0$	1817.569	$--$	$--$
∂h	578.407	0.000	-0.159	$\partial \beta_b^0$	0.000	$--$	$--$

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