Computing System Reliability Using Markov Chain Usage Models

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Abstract

Markov chains have been used successfully to model system use, generate tests, and compute statistics about anticipated system use in the field. Several reliability models are in use for Markov chain-based testing, but each has certain limitations. A Bayesian reliability model that is gaining support in field use is presented here.

1 Motivation

Statistical testing of systems based on a Markov chain usage model has proved itself to be a sound and cost-effective means to test systems (Kelly and Oshana 2000). Markov chain usage models provide a very effective means of test generation and a rich body of statistics useful in test planning and test analysis (Whittaker and Poore 1993; Whittaker and Thomason 1994; Prowell 2003).

Several reliability models have been proposed for use with Markov chain usage models.

- *Bernoulli sampling models*, which base the reliability estimate on the number of successful and unsuccessful tests. These do not take into account variation among tests, and assign equal weight to both long and short tests.

- *Failure state models*, which introduce special states in the model to account for failed transitions (Whittaker and Thomason 1994). These models typically do not provide the probability of failure when no failures are observed in testing, and make strong assumptions about state transition reliabilities.

- *Arc-based Bayesian model*, in which a reliability model (Miller, Morell, Noonan, Park, Nicol, Murriel, and Voas 1992) is applied to individual arcs of the Markov chain (Sayre 1999).

This paper focuses on the arc-based model. This reliability model was originally formulated as a simulation, and required the generation of many random test cases. Since each test case (depending on the usage model structure) could be very long, this approach could take considerable time to converge to a worthwhile estimate, even for smaller models (under 10,000 states). Since Markov chain usage models are seldom very large, an analytical solution for the reliability estimator and its variance is often faster and more precise than simulation. Investigation of an analytical solution for the estimator also reveals additional results, such as the overall probability of failure from any starting state.

2 Definitions

Let $P = [p_{i,j}]$ be the $n \times n$ transition matrix of a Markov chain usage model. Every such model has two special states: a *source* state, which is assumed to be the state with index one, and a *sink* state, which is assumed to be the state with index n, and the sole absorbing state. It is further assumed that all states are reachable from the source, and that there is a path to the sink from every state. Let Q denote the submatrix of P which omits the last row. For matrix A, let $\dot{A}$ denote the submatrix obtained by omitting the last column of A. Thus $\dot{Q}$ denotes the square transient submatrix of the Markov chain, and $(I - \dot{Q})^{-1}$ is the fundamental matrix for absorbing chains (Kemmeny and Snell 1976).

Let $r_{i,j}$ be a random variable called the *transition reliability*. This random variable counts the fraction of successful transitions from state i to state j. For the purpose of this paper, we assume the $r_{i,j}$ are governed by the beta distribution with parameters $a_{i,j}$ and $b_{i,j}$. In terms of the testing record, $a_{i,j} - 1$ is the number of successful transitions from state i to state j, and $b_{i,j} - 1$ is the number of failed transitions from state i to state j. Given this information, the moments $E[r_{i,j}^m]$ can be easily computed. Let $R_m = \left[E[r_{i,j}^m]\right]$ be the matrices of expectations, and let $V_m = \left[Var[r_{i,j}^m]\right]$ be the matrices of variances. Let $f_{i,j}$ be another random variable called the *transition failure rate*, defined as $f_{i,j} = 1 - r_{i,j}$. Note that $E[f_{i,j}] = 1 - E[r_{i,j}]$ and $Var[f_{i,j}] = Var[r_{i,j}]$. Let $F_m = \left[E[f_{i,j}^m]\right]$.

To shorten the equations which follow, let $\mathcal{F}_n$ denote the component-wise product of Q by F_n, and let $\mathcal{R}_n$ denote the component-wise product of Q by R_n. Dots may be applied: $\dot{\mathcal{F}}_n$ is the component-wise product of $\dot{Q}$ and $\dot{F}_n$.

The *single-use reliability* is defined as the probability that one successfully executes a "use" of the system under test from a defined start condition to a defined finish condition without observing a failure. In terms of the Markov chain usage model, it is the probability that one can move from the source state to absorption in the sink state without encountering any failures. The *single-use failure rate* is defined as one minus the single use reliability.

The following is taken from (Kemmeny and Snell 1976).

Theorem 1 Let A be an arbitrary matrix with $\lim_{m\to\infty} A^m = 0$. Then $I - A$ is non-singular and further:

$$\lim_{m\to\infty} \sum_{i=0}^{m} A^i = (I - A)^{-1}.$$

The following result will occasionally be necessary in order to apply the previous theorem.

Theorem 2 Let $A = [a_{i,j}]$ be a square matrix whose elements are all such that $0 \le a_{i,j} < 1$. Further, assume that each row of A sums to less than one. Then $\lim_{n\to\infty} A^n = 0$.

PROOF. The matrix A corresponds to the transient portion of a Markov chain transition matrix. Thus higher powers of the matrix vanish, since absorption is certain.

3 Single-Use Reliability Estimator

One could try to solve for the single-use reliability as follows. Let f_i^* be a random variable counting the fraction of times one experiences a failure prior to reaching the sink, given that one starts in state i. Then one can use the method of first passage to obtain:

$$f_i^* = \sum_k p_{i,k} f_{i,k} + \sum_k p_{i,k}(1 - f_{i,k}) f_k^*.$$

A naive approach would be to replace the $f_{i,j}$ with their expectations in order to compute $E[f_i^*]$. Unfortunately, this will not work for most models because the f_i^* are not independent (Sayre 1999). To avoid this problem, the derivations here will use a different approach and make use of the integral forms of expectation and variance.

Let $f_i^{(m)}$ be a random variable counting the fraction of times that a failure is observed on the mth step of a realization starting in state i, with all prior transitions successful. Clearly $E[f_i^{(1)}] = \sum_j p_{i,j} E[f_{i,j}]$.

Let T_m denote the set of all trajectories of length $m \geq 1$ which originate in state s_1. Each $t \in T_m$ visits states $s_1, s_2, \ldots, s_{m+1}$, in order. Of these, only s_{m+1} may be the sink. Let $F(t)$ denote the probability that trajectory t executes without failure up to the last step, which then fails. This gives the following equation.

$$E[f_{s_1}^{(m)}] = \sum_{t \in T_m} \Pr[t] \int_0^1 \Pr[F(t) = f] \, f \, \mathrm{d}f.$$

This sum can be re-written as the sum over all state sequences $s_1, s_2, \ldots, s_{m+1}$, factored and re-organized, with the following expression obtained for $m \geq 2$.

$$\begin{aligned} E[f_{s_1}^{(m)}] &= \sum_{s_2} p_{s_1,s_2} \int_0^1 \Pr[r_{s_1,s_2} = r_1] r_1 \, E[f_{s_2}^{(m-1)}] \, \mathrm{d}r_1 \\ &= \sum_{s_2} p_{s_1,s_2} E[r_{s_1,s_2}] \, E[f_{s_2}^{(m-1)}] \, \mathrm{d}r_1. \end{aligned} \tag{1}$$

Let U be a vector of ones of the appropriate size. Combining equation 1 with the expression for $E[f_i^{(1)}]$ given earlier, and placing the result in matrix-vector form, gives an expression for the vector $F^{(m)} = \left[E[f_i^{(m)}] \right]$ for any $m \geq 1$.

$$F^{(m)} = \begin{cases} \mathcal{F}_1 U & \text{if } m = 1 \\ \dot{\mathcal{R}}_1 F^{(m-1)} & \text{if } m > 1. \end{cases} \tag{2}$$

The use of the dot in equation 2 is to exclude the sink on all but the last step. The following result can be shown based on equation 2 for $m \geq 1$ by induction on m.

$$F^{(m)} = \dot{\mathcal{R}}_1^{m-1} \mathcal{F}_1 U. \tag{3}$$

Let $F^* = [f_i^*]$ be the vector of single-use failure rates. Using equation 3 and theorems 1 and 2 gives the following.

$$\begin{aligned} F^* &= \sum_{m=1}^{\infty} F^{(m)} \\ &= (I - \dot{\mathcal{R}}_1)^{-1} \mathcal{F}_1 U. \end{aligned} \tag{4}$$

Equation 4 provides a direct means to compute the expected single-use failure rates for all states, including the source.

4 Single-Use Reliability Variance

The variance of $f_i^{(m)}$ requires computation of the second moment of $f_i^{(m)}$. Following the pattern of the last section gives the following.

$$F_2^{(m)} = \dot{\mathcal{R}}_2^{m-1} \mathcal{F}_2 U. \tag{5}$$

The following equation can be derived for the variance of the single-use failure rate.

$$E\left[(f_i^*)^2\right] = \sum_{j=1}^{\infty} E\left[(f_i^{(j)})^2\right] + 2 \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} E\left[f_i^{(m)} f_i^{(n)}\right]. \tag{6}$$

The significant term in equation 6 is the last expectation. Again, by looking at the term over all trajectories and factoring, the following is obtained, where $F^{(m,n)} = \left[E\left[f_i^{(m)} f_i^{(n)}\right]\right]$.

$$F^{(m,n)} = \begin{cases} (\dot{\mathcal{R}}_1 - \dot{\mathcal{R}}_2) F^{(n-1)} & \text{if } m = 1 \\ \dot{\mathcal{R}}_2 F^{(m-1,n-1)} & \text{if } m > 1. \end{cases} \tag{7}$$

Using induction and equation 7 one can show the following for $n > m \geq 1$.

$$F^{(m,n)} = \dot{\mathcal{R}}_2^{m-1} (\dot{\mathcal{R}}_1 - \dot{\mathcal{R}}_2) \dot{\mathcal{R}}_1^{n-m-1} \mathcal{F}_1 U. \tag{8}$$

Combining equation 8 with equation 6 one can obtain the vector of second moments F_2^*.

$$F_2^* = (I - \dot{\mathcal{R}}_2)^{-1} \mathcal{F}_2 U + 2(I - \dot{\mathcal{R}}_2)^{-1} (\dot{\mathcal{R}}_1 - \dot{\mathcal{R}}_2) F^*.$$

This allows the variances to be obtained.

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