TECHNOMETRICS, MAY 1988, VOL. 30, NO. 2

Bayesian Reliability Analysis of Series Systems of Binomial Subsystems and Components

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A Bayesian procedure is presented for estimating the reliability of a series system of independent binomial subsystems and components. The method considers either test or prior data (perhaps both or neither) at the system, subsystem, and component level. Beta prior distributions are assumed throughout. Inconsistent prior judgments are averaged within the simple-to-use procedure. The method is motivated by the following practical problem. It is required to estimate the overall reliability of a certain air-to-air heat-seeking missile system containing five major subsystems with up to nine components per subsystem. The posterior distribution of the overall missile-system reliability from which the required estimates are obtained is computed.

KEY WORDS: Approximate prior; Bayesian probability bound; Binomial sampling; Beta prior.

1. INTRODUCTION

The problem of obtaining Bayesian estimates of the reliability of a series system of independent pass/fail (binomial) components is an important one. It was initially solved by Springer and Thompson (1966a, 1969) for the case of component beta prior distributions. The resulting posterior-system reliability distribution is found by evaluating the inversion integral of the product of the Mellin transforms of the component posterior beta distributions (Springer 1979). The Springer and Thompson approach uses binomial-test and beta-prior data for each component.

Mastran (1976) and Mastran and Singpurwalla (1978) extended the Bayesian procedure to permit the reliability assessment of a coherent system using test and prior data at both the component and system levels. Cole (1975) also described a Bayesian procedure for integrating component and system-level test and prior data. Dostal and Iannuzzelli (1977) proposed a similar method but did not assume that system-test data are available. An excellent survey of Bayesian interval estimation methods for the reliability of various systems was presented by Thompson and Haynes (1980).

Some real-world series systems have a more complicated data structure than any of the existing models can accommodate. In some cases, there may exist test or prior data at the system, subsystem, and component levels. There may be even more than three configuration levels in the series system with either test or prior data at each level. Only three levels are considered here, however.

The Bayesian model to be presented was motivated by the following problem. It was required to estimate the overall reliability of a certain air-to-air heat-seeking missile system under specific use conditions. The system as represented here consists of the five major subsystems and associated components given in Table 1. The numbers in parentheses are subsystem and component identifiers, which will be considered later. Note that the components in the warhead are coded A–I to avoid problems with security classification. The test and prior data for each of the components and subsystems, as well as the overall system, are presented in Section 3. The problem is to combine all of these data to estimate the reliability of the overall missile system in its operational context. Inconsistencies in prior judgments will be averaged by the analyst as described in Section 2.

The preceding system may be generally described as one in which there are m independent subsystems in series in which the ith subsystem contains k_i independent components in series. There exist either binomial test or prior data (possibly both or perhaps neither) on the components, subsystems, and system.

Table 1. The Major Subsystems and Components of a Certain Air-to-Air Heat-Seeking Missile System

Warhead (1)	*Missile (2)*	*Aircraft (3)*	*C³I (4)*	*Logistics/ Maintenance (5)*
A(11)	Power supply (21)	Flight structure (31)	Airspace control effectiveness (41)	Ground handling (51)
B(12)	Target acquisition/guidance system (22)	Avionics (32)	Rules of engagement (42)	Storage (52)
C(13)	Motor (23)	Power (33)	IFF/visual (43)	Missile availability (53)
D(14)	Flight structure (24)	Flight control (34)	Aircraft on-station availability (44)	
E(15)	Aircraft interface (25)	Environmental (35)	Radio communications (45)	
F(16)	Control (26)	Acquisition/fire control (36)		
G(17)		Launching (37)		
H(18)		Missile interface (38)		
I(19)		Human intervention (39)		

NOTE: Headings denote subsystems. IFF means identification friend or foe.

The problem is to statistically use all of these data to make inferences about the overall system reliability; specifically, both point and interval estimates are desired. These inferences were to be used as figures of merit for examining the reliability impact of additional testing within the overall system.

The distinguishing features of our approach are the following:

1. the statistical consideration and simultaneous use of test or prior data at the component, subsystem, and system level of a series system.
2. the development of a method for an interactive analyst to average inconsistent prior judgments at the subsystem and system levels.
3. the use of some key approximations (the accuracy of which is examined as part of the method) that provide a simple and easily applied closed-form solution to the problem.

In Section 2 we present a proposed solution to the problem. The missile-system problem is again considered in Section 3, where it is used to illustrate the method. Some conclusions are presented in Section 4.

2. A TWO-STAGE BAYESIAN MODEL

A binomial model describes the number of survivors s in n independent tests; the outcomes, success or failure, are statistically independent for each test; and the reliability (the survival probability) r is constant across the tests. If these conditions are true, then the number of survivors follows the well-known binomial distribution given by

$$f(s \mid r; n) = \frac{n!}{(n-s)!s!} r^s(1-r)^{n-s},$$

$$s = 0, 1, \ldots, n, \quad 0 \le r \le 1. \quad (1)$$

A widely used prior distribution for r is provided by the beta distribution, with pdf given by

$$g(r; p, q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} r^{p-1}(1-r)^{q-1},$$

$$0 \le r \le 1, \quad p, q > 0, \quad (2)$$

which will henceforth be denoted by a $\beta(p, q)$ distribution. What about the assumption of a beta prior distribution? Weiler (1965) concluded that the effect of assuming a beta prior in binomial sampling, when in fact the true prior is not a beta distribution, is negligible in many practical applications. It is also known that the beta family is rich in shapes. In addition, Dyer and Chiou (1984) found that only the beta family gives a most conservative prior distribution with specified mean (where conservatism refers to restraint of extraneous information that is embedded in any prior) from among nine broad families of priors. They also concluded that the beta family is the best

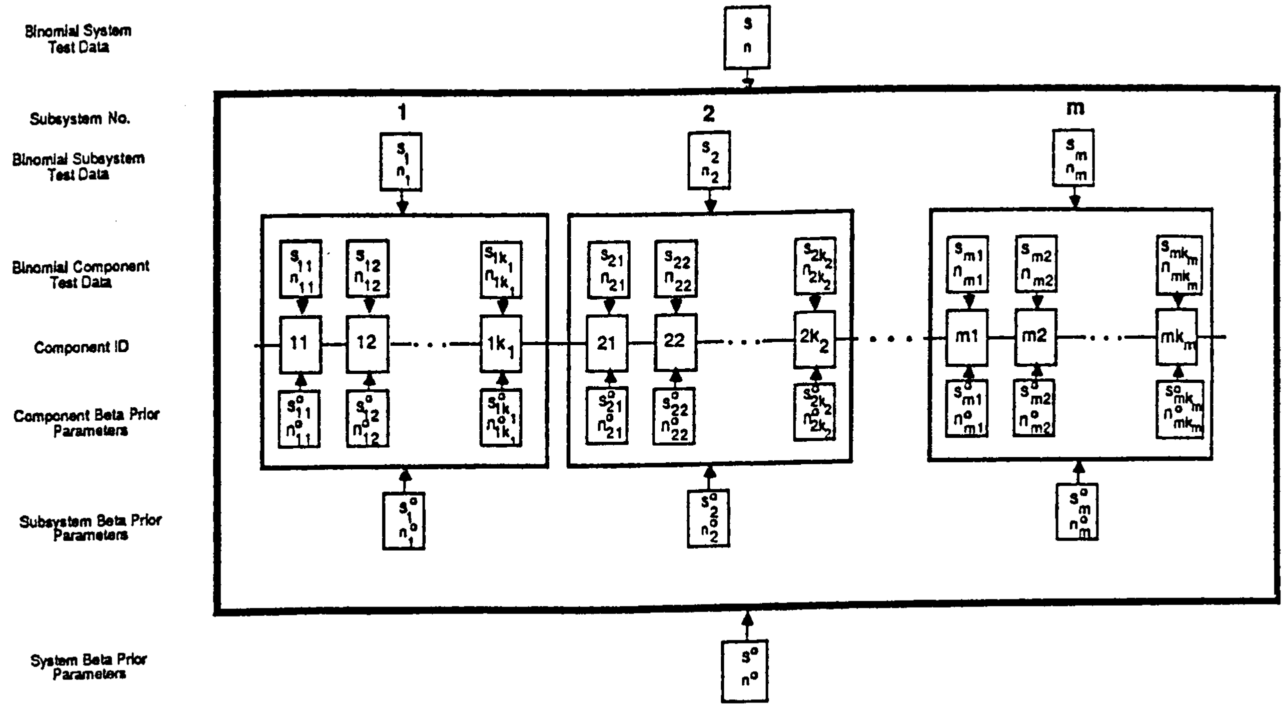

Figure 1. Series System Configuration, Notation, and Data Structure.

choice for use in determining the most conservative prior when there is no available prior information about the reliability in question. Colombo and Constantini (1981) also provided a rational reconstruction of the beta prior distribution in binomial sampling.

The mean and variance of (2) are $p/(p+q)$ and $pq/[(p+q)^2(p+q+1)]$, respectively. Mosleh and Apostolakis (1982) described a simple procedure for identifying values for p and q for specified moments and percentiles. Their method is particularly appropriate for high-reliability devices in which the beta distributions are concentrated near 1.

Figure 1 shows the system configuration, notation, and data structure considered here. Note that a double subscript denotes test and prior data at the component level, a single subscript denotes data at the subsystem level, and no subscript designates system-level data. Prior beta distribution parameters are identified with the superscript *o*, and the absence of such a superscript denotes binomial test results. The component identifiers and corresponding data subscripts contain two terms, i and j; i identifies the subsystem, and j denotes the component within the subsystem.

Consider the component, subsystem, and system beta prior distributions and parameters. At the component level a $\beta(s_{ij}^o+1,\ n_{ij}^o-s_{ij}^o+1)$ prior distribution is used in which, for the case of nonnegative integers, s_{ij}^o may be interpreted as one less than the number of prior component successes and n_{ij}^o is two less than the number of prior component tests. This beta parameterization is the form considered by Springer and Thompson (1966a,b, 1969) and their results, described in Section 2.3, are directly applicable. For example, $n_{ij}^o = s_{ij}^o = 0$ is interpreted as one prior success in two prior tests and corresponds to a $\beta(1, 1)$ or uniform prior. In the case of noninteger values of s_{ij}^o or n_{ij}^o, no such interpretation is possible; in this case, s_{ij}^o and n_{ij}^o simply denote component prior beta parameters. For example, $n_{ij}^o = -1$, $s_{ij}^o = -\frac{1}{2}$ yields Jeffreys's noninformative prior (Box and Tiao 1973), which is also known as the arcsine distribution. In the case of subsystem and system beta prior distributions, $\beta(s_i^o+1,\ n_i^o-s_i^o+1)$ and $\beta(s^o+1,\ n^o-s^o+1)$, respectively, the parameters are restricted to being nonnegative integers (for reasons presented later) and the previous interpretation holds.

The *induced prior distribution* for a given subsystem is the posterior distribution based on all component prior and test data in a series configuration. In the same way the subsystem posterior distributions together induce a prior at the system level. In addition to these induced priors, additional prior data in the form of (s_i^o, n_i^o) or (s^o, n^o) that are not a function of the lower-order data in the system may exist. The corresponding $\beta(s_i^o+1,\ n_i^o-s_i^o+1)$ and $\beta(s^o+1,\ n^o-s^o+1)$ priors are referred to here as *native prior distributions.*

Figure 2 presents the procedural steps of a two-stage Bayesian solution to the problem. The steps in the procedure will now be described.

Bayesian Analysis Model for Series Systems

Stage 1 - Subsystem-Level Analysis

- Step 1: Compute the Component Posterior Beta Distributions
- Step 2: Compute the Induced Subsystem Prior Approximate Beta Distribution
- Step 3: Examine the Quality of the Beta Approximation in Step 2
- Step 4: Weight and Combine the Native and Induced Subsystem Prior Beta Distributions
- Step 5: Compute the Subsystem Posterior Beta Distribution

(Repeat Steps 1-5 for Each Subsystem)

Stage 2 - System-Level Analysis

- Step 6: Compute the Induced System Prior Approximate Beta Distribution
- Step 7: Examine the Quality of the Beta Approximation in Step 6
- Step 8: Weight and Combine the Native and Induced System Prior Beta Distributions
- Step 9: Compute the Final System Posterior Beta Distribution

Figure 2. The Steps in the Two-Stage Bayesian Procedure.

2.1 Step 1: Compute the Component Posterior Beta Distributions

Consider the jth component in the ith subsystem. For a binomial sampling model $f(s_{ij} \mid r_{ij}; n_{ij})$ and a $\beta(s_{ij}^o + 1, n_{ij}^o - s_{ij}^o + 1)$ prior on r_{ij}, the corresponding posterior distribution is a $\beta(s_{ij} + s_{ij}^o + 1, n_{ij} + n_{ij}^o - s_{ij} - s_{ij}^o + 1)$ distribution. Of course, if there are no binomial test data for a given component, then the posterior is the same as the prior distribution. In the absence of prior test results, some type of noninformative prior must be used, such as a $\beta(1, 1)$ or $\beta(.5, .5)$ distribution.

2.2 Step 2: Compute the Induced Subsystem Prior Appropriate Beta Distribution

The induced prior distribution for the reliability r_i of the ith subsystem, $g(r_i)$, is the distribution of the product of k_i independent beta random variables in which the jth variable has the posterior beta distribution given in Section 2.1. Note that, for notational ease, we have suppressed the conditional dependency of $g(r_i)$ on the component prior and test data, a convention that we will continue to follow. Although it is often possible to determine the exact distribution of this product (see Sec. 2.3), Thompson and Haynes (1980) suggested approximating the exact subsystem prior with a beta distribution having the same first two moments. Springer (1979) also mentioned such an approximation. Using this approximation, we compute the approximate induced prior distribution on r_i, denoted by $g_a(r_i)$, as a $\beta(a_i, b_i)$ distribution in which

$$a_i = [M_i^2(1 - M_i) - V_i M_i]/V_i$$
$$b_i = [M_i(1 - M_i)^2 - V_i(1 - M_i)]/V_i \qquad (3)$$

and

$$M_i = \prod_{j=1}^{k_i} \left[\frac{s_{ij}^o + s_{ij} + 1}{n_{ij}^o + n_{ij} + 2}\right]$$
$$V_i = \prod_{j=1}^{k_i} \left[\frac{(s_{ij}^o + s_{ij} + 1)(s_{ij}^o + s_{ij} + 2)}{(n_{ij}^o + n_{ij} + 2)(n_{ij}^o + n_{ij} + 3)}\right] - M_i^2. \qquad (4)$$

The corresponding approximate induced prior beta cdf on r_i will be denoted by $G_a(r_i)$.

2.3 Step 3: Examine the Quality of the Beta Approximation in Step 2

The exact induced subsystem prior density $g(r_i)$ can be obtained in closed form for the case in which s_{ij}^o and n_{ij}^o are nonnegative integers. Springer and Thompson (1966a) obtained the solution by expanding the Mellin transform of the induced subsystem prior in partial fractions and then applying the Mellin inversion integral (Springer 1979, p. 96). Springer and Thompson (1969) provided a FORTRAN computer program for performing the necessary computations. For example, using this program for $k_i = 3$ and component data $n_{i1}^o + n_{i1} = s_{i1}^o + s_{i1} = 0$, $n_{i2}^o + n_{i2} = 9$, $s_{i2}^o + s_{i2} = 7$, $n_{i3}^o + n_{i3} = 4$, and $s_{i3}^o + s_{i3} = 3$ yields the exact induced ith subsystem prior density function $g(r_i) = 2.38 - 40.00\, r_i^3 + 60.00\, r_i^4 - 85.71\, r_i^7 + 90.00\, r_i^8 - 26.67\, r_i^9$ $(0 < r_i < 1)$. A few other examples may be found in Springer and Thompson (1966a,b, 1969) and Springer (1979).

Unfortunately, the FORTRAN code is quite computationally unstable. Its use results in frequent numerical problems for practical data sets and multiple precision floating-point arithmetic only yields some improvement. We have found that these problems frequently occur with data indicative of highly reliable components, which is often the case in practice. Thus we use Monte Carlo simulation to approximate the exact-induced ith subsystem prior distribution. For convenience, the simulation-produced exact-induced ith subsystem prior empirical distribution function will be denoted by $G(r_i)$.

Thus in Step 3 we examine the quality of the beta approximation to the exact-induced ith subsystem prior distribution by simulating the distribution of the product $r_i = \sum_{j=1}^{k_i} r_{ij}$, where r_{ij} has a $\beta(s_{ij} + s_{ij}^o + 1, n_{ij} + n_{ij}^o - s_{ij} - s_{ij}^o + 1)$ distribution. We then examine the quality of the beta approximation by computing the Kolmogorov–Smirnov (KS) two-sided p value for the beta hypothesis (Conover 1980). The empirical cdf for this calculation is based on a sample of 100 observations of the product r_i. The assumption of a beta-induced subsystem prior cannot be rejected at a significance level less than or equal to this p value. In other words, the larger the p value the better the assumption of a beta-induced prior approximation. For small p values, the beta approximation is unacceptable. In this case, the procedure may be continued using a nonparameteric density function estimate of the exact distribution; the density estimate should be based on a large Monte Carlo sample of, say, 1,000 observations to ensure good convergence of the estimate. This procedure is numerically much more difficult than using the beta approximation, however. Finally, further qualitative assurance of the appropriateness of the beta approximation may be gained by visually comparing plots of $G_a(r_i)$ and $G(r_i)$, as well as selected quantiles of both distributions.

2.4 Step 4: Weight and Combine the Native and Induced Subsystem Prior Beta Distributions

The ith subsystem native $\beta(s_i^o + 1, n_i^o - s_i^o + 1)$ and induced $\beta(a_i, b_i)$ priors may be averaged to produce a combined single beta prior by means of a natural-conjugate (NC) method proposed by Winkler (1968). In this method, Bayes's theorem is used to determine a combined prior by interpreting the native beta prior as being proportional to a binomial likelihood. If s_i^o and n_i^o are nonnegative integers, then the native prior is interpreted as being proportional to the likelihood given $s_i^o + 1$ prior successes in $n_i^o + 2$ prior tests. Weighting the native and induced priors, and combining them through Bayes's theorem, yields the ith subsystem combined prior $\beta(w_{i1}a_i + w_{i2}s_i^o + w_{i2}, w_{i1}b_i + w_{i2}n_i^o - w_{i2}s_i^o + w_{i2})$ distribution. If the beta approximation is unsatisfactory in Step 3, then a weighted average of the native beta-induced and exact-induced subsystem priors is suggested as a means for obtaining a combined prior distribution.

There is a twofold problem in determining the weights w_{i1} and w_{i2}. Not only must the relative weights for both the native and induced priors be selected, but the sum of the weights must also be determined, as there is no mathematical restriction on this sum. Winkler (1968) argued that $1 \le w_{i1} + w_{i2} \le 2$. Furthermore, the sum should approach 2 when the native-prior and induced-prior information may be considered to be reasonably independent and should approach 1 when the prior information appears to be dependent (in the sense that the two priors are based at least in part on the same information). The latter condition is probably more realistic here than the former condition, because often in practice the same system analyst (or group of analysts) is asked to assess both the ith subsystem native beta prior and the component beta priors in the ith subsystem. Unless the two priors really do embody somewhat independent sets of information, setting the sum much greater than 1 may result in spurious variance reduction.

If it becomes difficult for the analyst to uniquely identify the weights to be used, then the sensitivity of the final system-reliability distribution to different weight choices can be examined. If this distribution is insensitive to these choices, then it does not matter; otherwise, either an effective range of final reliability estimates can be reported or additional effort can be applied to identify the unique choice to be considered.

We chose the NC method here because the combined prior is always a beta distribution (this is a requirement for Stage 2 of the Bayesian procedure). Although there exist many other averaging methods (such as taking a weighted average of the two priors), the use of these methods does not guarantee a beta-combined prior.

Moreover, if there is no native prior data on the ith subsystem, then we set $w_{i1} = 1$ and $w_{i2} = 0$ and the combined prior becomes the induced prior. On the other hand, a system analysis can be forced to begin at the subsystem level by ignoring the induced subsystem prior ($w_{i1} = 0$, $w_{i2} = 1$). Actual performance of the NC method will be illustrated in Section 3.

2.5 Step 5: Compute the Subsystem Posterior Beta Distribution

Combining the binomial test data (s_i, n_i) and the combined beta prior using Bayes's theorem yields the ith subsystem posterior $\beta(w_{i1}a_i + w_{i2}s_i^o + s_i + w_{i2}, w_{i1}b_i + w_{i2}n_i^o - w_{i2}s_i^o + n_i - s_i + w_{i2})$ distribution.

After the subsystem posterior beta distributions have been computed for all m subsystems, the system-level analysis (Stage 2) proceeds as follows.

2.6 Step 6: Compute the Induced System Prior Approximate Beta Distribution

The induced prior density $g(r)$ for the overall system reliability r is the distribution of the product of m independent beta random variables, in which the ith variable has the subsystem posterior beta distribution given in Section 2.5. As in Step 2, we again approximate the exact distribution of this product using a beta distribution having the same first two moments. Thus we compute the approximate induced prior density on r, denoted by $g_a(r)$, as a $\beta(a, b)$ distribution. The parameters a and b are computed from (3) by replacing M_i and V_i in (4) with M and V, respectively, where

$$M = \prod_{i=1}^{m} \left[\frac{w_{i1}a_i + w_{i2}s_i^o + s_i + w_{i2}}{w_{i1}a_i + w_{i1}b_i + w_{i2}n_i^o + n_i + 2w_{i2}} \right] \quad (5)$$

and

$$V = \prod_{i=1}^{m} \left[\frac{(w_{i1}a_i + w_{i2}s_i^o + s_i + w_{i2})(w_{i1}a_i + w_{i2}s_i^o + s_i + w_{i2} + 1)}{(w_{i1}a_i + w_{i2}b_i + w_{i2}n_i^o + n_i + 2w_{i2})(w_{i1}a_i + w_{i2}b_i + w_{i2}n_i^o + n_i + 2w_{i2} + 1)} \right] - M^2. \tag{6}$$

The corresponding approximate induced prior beta cdf on r is denoted by $G_a(r)$.

2.7 Step 7: Examine the Quality of the Beta Approximation in Step 6

It is difficult to obtain the exact analytical distribution of the product of the m independent subsystem posterior beta distributions, because all of the respective beta parameters are extremely unlikely to be nonnegative integers. Because it is easy to generate beta pseudorandom variates having noninteger parameters, however, Monte Carlo simulation is recommended as a way to approximate this exact distribution (in the manner described in Sec. 2.3). The simulation-produced induced system prior empirical distribution function will be denoted by $G(r)$.

We then examine the quality of the beta approximation to the exact distribution by again computing the KS two-sided p value as in Step 3. The empirical cdf for computing this statistic is again based on a Monte Carlo sample of size 100. As before, the beta-distribution approximation is considered to be unacceptable if the p value is small. In this case the procedure could be continued using a nonparametric density function estimate of the exact induced system prior instead of the beta approximation. To ensure good convergence of the density estimate, a large Monte Carlo sample should be used in computing the density estimate. As in Step 3, additional qualitative assurance of the beta approximation's suitability may be gained by comparing plots of $G_a(r)$ and $G(r)$ as well as selected quantiles of both distributions.

2.8 Step 8: Weight and Combine the Native and Induced System Prior Beta Distributions

The NC method described in Section 2.4 is again used to average the native $\beta(s^o + 1, n^o - s^o + 1)$ and induced $\beta(a, b)$ priors to produce a combined system beta prior. Following the NC procedure, and constraining s^o and n^o to nonnegative integers, yields the combined system prior $\beta(w_1 a + w_2 s^o + w_2,\ w_1 b + w_2 n^o - w_2 s^o + w_2)$, where w_1 is the weight of the induced system prior and w_2 is the weight applied to the native prior. As previously discussed, $1 \le w_1 + w_2 \le 2$, and both the sum of the weights and the individual weights are again chosen based on the discussion in Section 2.4. This will also be illustrated in Section 3. If the beta approximation is unsatisfactory in Step 7, then a weighted average of the native beta and the nonparametric density function estimate of the induced system prior is again suggested as a method for obtaining a combined distribution.

2.9 Step 9: Compute the Final System Posterior Beta Distribution

Combining the binomial system test data (s, n) and the combined system beta prior using Bayes's theorem yields the final system posterior $\beta(w_1 a + w_2 s^o + s + w_2,\ w_1 b + w_2 n^o - w_2 s^o + n - s + w_2)$ distribution of the system reliability r. Desired inferences about the system reliability are then obtained from this distribution in the usual way; for example, the posterior mean is the Bayesian point estimate of r for a squared-error loss function, whereas the .05 and .95 quantiles define a 90% Bayesian probability interval estimate for r.

3. EXAMPLE

As indicated in Section 1, the example considered here concerns the overall reliability of a certain air-to-air heat-seeking missile system. Table 2 contains the binomial test and beta prior data for the components and subsystems identified in Table 1, as well as those for the overall system. Some of the data in this example have been deliberately altered to avoid security classification problems. Corresponding blanks indicate that either the prior or the test data (or both) are missing.

If no prior data exist for a given component, then some type of noninformative prior must be considered. For this example we have chosen to use Jeffreys's prior in Step 1 for Subsystem 1. On the other hand, if prior data are missing at the subsystem level (such as Subsystems 1, 2, and 5), then there is no need to even consider the corresponding native prior, because no weight will be given to this native prior in Step 4.

Consider the subsystem-level analysis, Stage 1. The induced prior approximate distribution for the warhead reliability, $g_a(r_1)$, is computed using Equations (3) and (4) in Step 2 to be a $\beta(10.64, 3.51)$ distribution having .05, .5, and .95 quantiles of .54, .76, and .91, respectively. The KS p value of .81 indicates that this beta approximation is quite satisfactory.

Because of the lack of native prior data on Subsystem 1, the combined prior is the induced prior distribution; thus $w_{11} = 1$ and $w_{12} = 0$ in Step 4. Figure 3 gives a plot of the induced prior and corresponding posterior $\beta(18.64, 3.51)$ distributions of the warhead reliability (Step 5). The posterior reflects the increase in degree of belief resulting from the eight successful

Table 2. Component, Subsystem, and System Binomial Test and Beta Prior Data for a Certain Air-to-Air Heat-Seeking Missile System

i	j	s°_{ij}	n°_{ij}	s_{ij}	n_{ij}	s°_{i}	n°_{i}	s_i	n_i	s°	n°	s	n
										system 115	system 265		
1								8	8				
1	1			30	30								
1	2			80	80								
1	3			39	40								
1	4			30	30								
1	5	845	846	90	90								
1	6			10	10								
1	7			29	30								
1	8			20	20								
1	9			5	5								
2								7	8				
2	1	398	400	50	50								
2	2	277	300	50	50								
2	3	1,097	1,100	99	100								
2	4	653	688	23	25								
2	5	298	300	50	50								
2	6	347	350	55	55								
3						257	269	191	205				
3	1	245	248	129	130								
3	2	244	248	130	130								
3	3	246	248	129	130								
3	4	271	274	129	130								
3	5	356	358	130	130								
3	6	253	255	247	250								
3	7	249	250	129	130								
3	8	249	250	249	250								
3	9	340	350	330	330								
4						55	66						
4	1	796	800										
4	2	795	800										
4	3	793	800										
4	4	790	800										
4	5	385	400										
5													
5	1	1,025	1,120										
5	2	1,086	1,090										
5	3	1,083	1,090										

warhead tests. The .05, .5, and .95 quantiles of the posterior distribution are .70, .85, and .95, respectively.

Now consider the missile, Subsystem 2. The induced prior approximate distribution for the missile reliability is a $\beta(405.11, 67.68)$ distribution. A corresponding KS p value of .20 indicates that the approximation is again satisfactory and the .05, .5, and .95 quantiles are .83, .86, and .88, respectively.

The combined prior in Step 4 is again the induced prior beta distribution. Because of this strong prior and compatible test data, the seven successful missile tests in eight trials have no noticeable effect on the degree of belief regarding the missile reliability. Figure 4 gives a plot of the induced prior and posterior $\beta(412.11, 68.68)$ distributions for the missile reliability. The two distributions essentially overlay each other.

For the reliability of Subsystem 3, the aircraft, the induced prior distribution is computed in Step 2 to be a $\beta(419.46, 43.03)$ distribution having an associated KS p value of .70. Now let us compute the combined prior in Step 4. The native prior data for the aircraft is assumed to be nonindependent of the aircraft component prior data, as the same system analyst supplied both sets of prior data. Accordingly, the sum of the weights was taken to be 1. Note that the native prior $\beta(258, 13)$ distribution reflects a precise state of increased optimism regarding the aircraft reliability compared with the corresponding induced prior. Neither prior was believed to be superior and thus equal weights were chosen in Step 4; however, the sensitivity of the results to this choice will be considered later. This choice yielded a $\beta(338.73, 28.01)$ combined prior, which is plotted in Figure 5 along with the induced and native priors. Note that

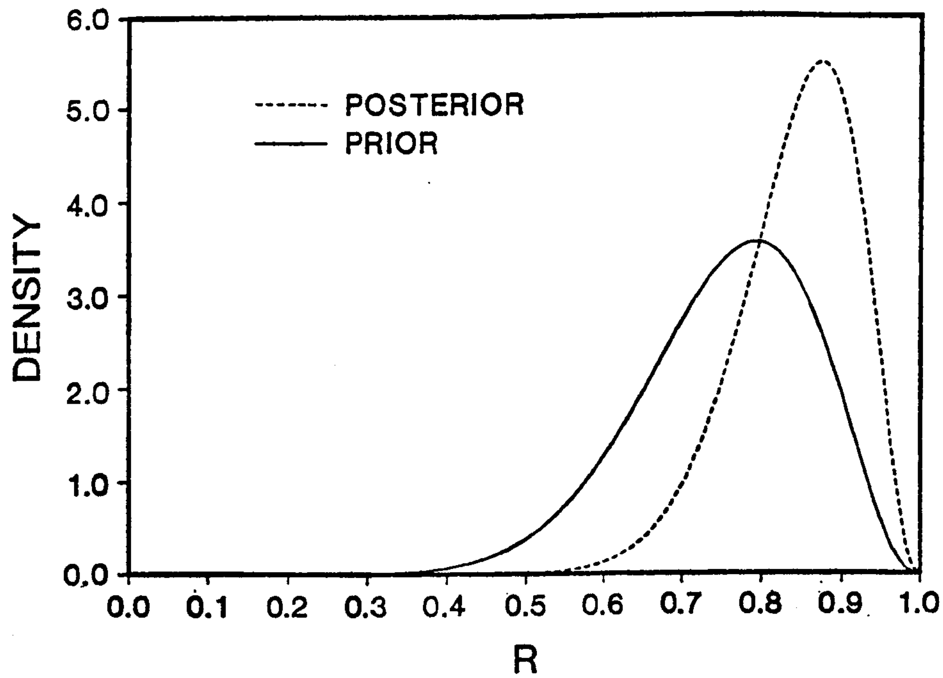

Figure 3. The Prior and Posterior Beta Warhead Reliability Distributions.

the combined prior is located nearly midway between the two priors, while having only a slightly larger spread. The combined prior and corresponding posterior $\beta(529.73, 42.01)$ aircraft-reliability distributions are plotted in Figure 6. The .05, .5, and .95 prior and posterior quantiles are .90, .92, .95, and .91, .93, .94, respectively. The effect of the 191 successes of 205 aircraft tests is only slight because of the compatibility of the test and prior data.

The induced prior approximate distribution for the C^3I subsystem reliability is computed to be a $\beta(500.88, 40.75)$ distribution having an associated p value of .99. The .05, .5, and .95 quantiles are .91, .93, and .94, respectively. The native $\beta(56, 12)$ prior distribution for the C^3I subsystem reliability is more diffuse and is shifted to the left of the induced prior. The induced and native priors are plotted in Figure 7 along with the combined prior using the weights

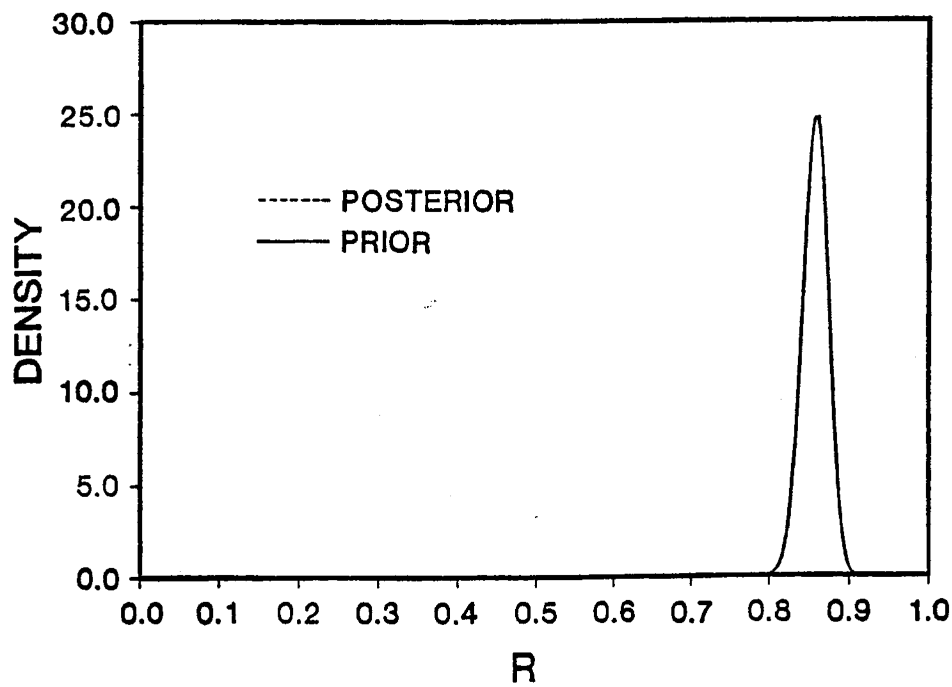

Figure 4. The Prior and Posterior Beta Missile Reliability Distributions.

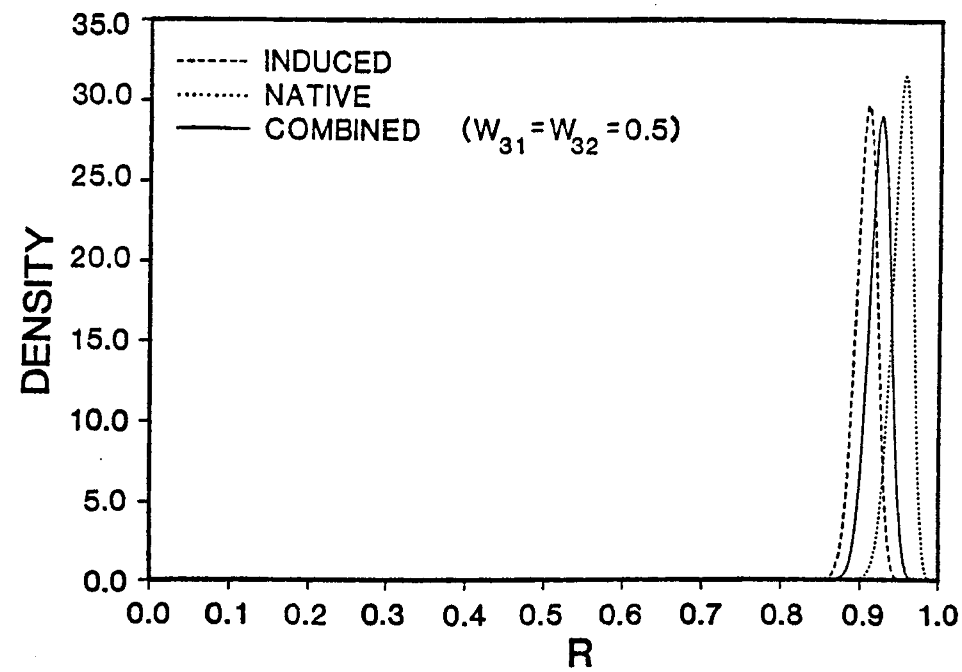

Figure 5. The Induced, Native, and Combined Prior Aircraft Reliability Distributions.

$w_{41} = .25$ and $w_{42} = .75$. Although these were the weights identified by the system analyst, the sensitivity of the final system-reliability distribution to this choice will be subsequently considered.

Because there have been no independent binomial tests of the C^3I subsystem, the posterior C^3I reliability distribution reduces to the combined $\beta(167.22, 19.19)$ prior in Figure 7. The .05, .5, and .95 quantiles are .86, .90, and .93, respectively.

The induced prior approximate distribution for the reliability of the logistics/maintenance subsystem is a $\beta(1{,}021.76, 109.03)$ distribution with a KS p value of .40. The .05, .5, and .95 quantiles are .89, .90, and .92, respectively. The combined prior distribution is the induced prior approximate beta distribution, and because there have been no independent binomial subsystem tests of this subsystem, the final subsystem posterior reliability distribution is the same beta distribution.

Now consider the system-level analysis, Stage 2. At

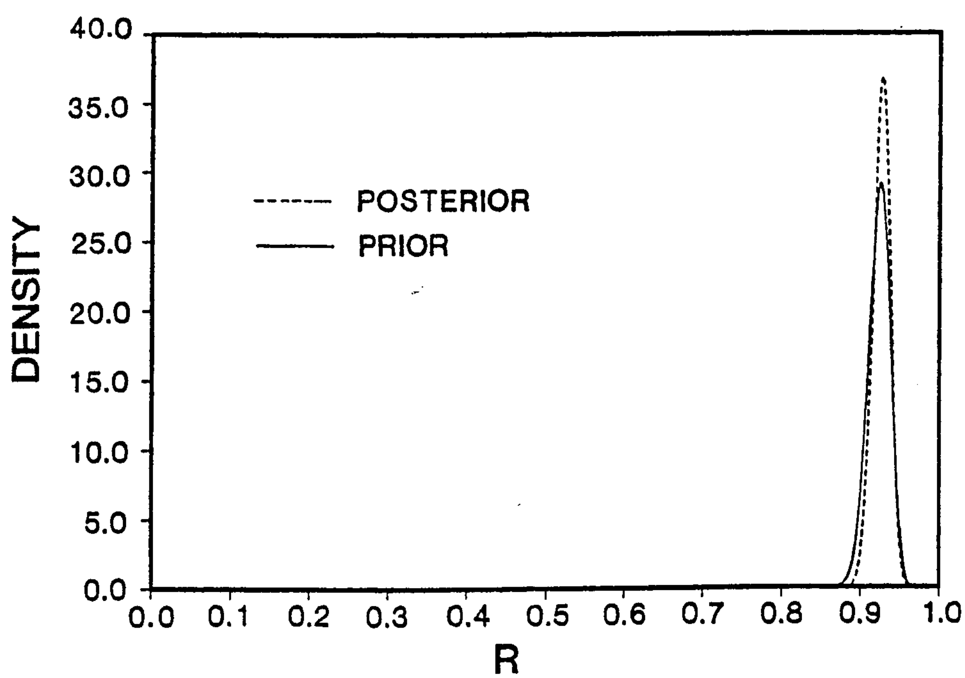

Figure 6. The Prior and Posterior Beta Aircraft Reliability Distributions.

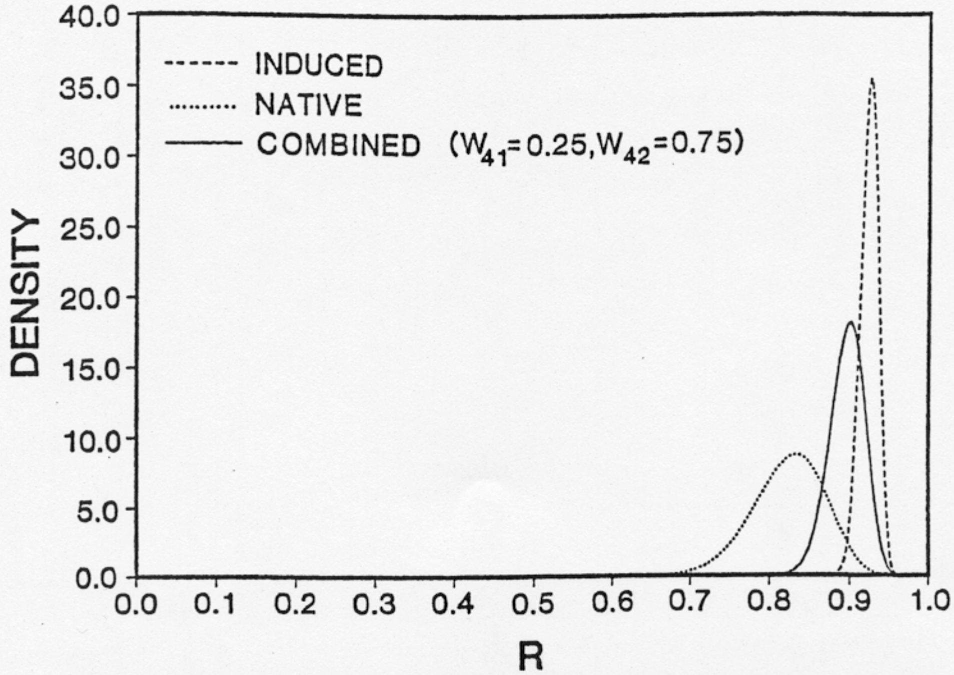

Figure 7. The Induced, Native, and Combined Prior C³I Reliability Distributions.

Step 6, the induced prior approximate system reliability distribution, $g_a(r)$, is computed, using Equations (3), (5), and (6), to be a $\beta(48.52, 41.06)$ distribution having an associated p value of .78. The .05, .5, and .95 quantiles of this distribution are .45, .54, and .63, respectively.

The native $\beta(116, 151)$ prior distribution for the overall system reliability is less diffuse and is shifted somewhat to the left of the induced prior. Neither prior was believed to be superior and thus $w_1 = w_2 = .5$ was chosen in Step 8. This yielded a $\beta(82.26, 96.03)$ combined prior, which is plotted in Figure 8 along with the induced and native system priors. Because there have been no overall system tests, the final posterior system reliability distribution in Step 9 is again a $\beta(82.26, 96.03)$ distribution. This narrow posterior is largely because of the strong prior data used in the analysis; weaker prior data would result in a more diffuse posterior.

A common point estimate of the overall reliability

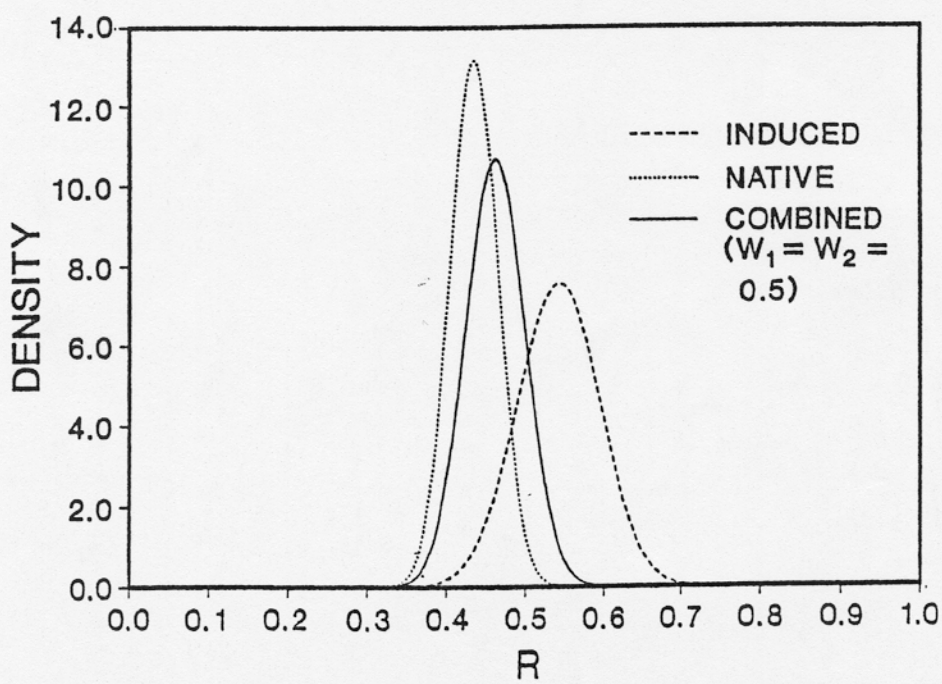

Figure 8. The Induced, Native, and Combined Prior System Reliability Distributions.

Table 3. Sensitivity of the Posterior System Reliability Distribution to Selected Choices of Weights Summing to 1

Case	w_{31}	w_{41}	w_1	Posterior distribution	Quantiles .05	.50	.95
1	.5	.25	.5	$\beta(82.26, 96.03)$	.40	.46	.52
2	0	.25	.5	$\beta(81.76, 94.83)$	.40	.46	.52
3	1	.25	.5	$\beta(82.63, 96.94)$	.40	.46	.52
4	.5	0	.5	$\beta(78.95, 96.68)$	.39	.45	.51
5	.5	1	.5	$\beta(82.60, 94.95)$	.40	.47	.53
6	—	—	0	$\beta(116, 151)$	.38	.43	.48
7	.5	.25	1	$\beta(48.52, 41.06)$	.45	.54	.63
8	.5	0	1	$\beta(41.90, 42.36)$	.41	.50	.59
9	.5	1	1	$\beta(49.19, 38.91)$	.47	.56	.64
10	0	.25	1	$\beta(47.52, 38.66)$	.46	.55	.64
11	1	.25	1	$\beta(49.27, 42.88)$	.45	.53	.62
12	0	0	.5	$\beta(78.59, 95.58)$	.39	.45	.51
13	1	0	.5	$\beta(79.22, 97.51)$	.39	.45	.51
14	0	1	.5	$\beta(82.06, 93.76)$	.41	.47	.53
15	1	1	.5	$\beta(83.00, 95.86)$	.40	.46	.53
16	1	1	1	$\beta(50.01, 40.72)$	.47	.55	.64

of this missile system is the posterior mean of .46, but a two-sided symmetric 90% Bayesian probability interval estimate for the unknown reliability is (.40, .52). Because of the near symmetry of the posterior distribution, the posterior median is also .46.

Suppose now that, instead of Jeffreys's $\beta(.5, .5)$ warhead component priors, we had used $\beta(1, 1)$ (uniform) priors for components 1–4 and 6–9. The posterior warhead reliability distribution now becomes $\beta(18.08, 6.19)$, having .05, .5, and .95 quantiles of .59, .75, and .88. These estimates are significantly less optimistic than those (.70, .85, .95) obtained using Jeffreys's priors, as $\beta(.5, .5)$ is more diffuse than $\beta(1, 1)$, and this has an effect when few (if any) failures occur on components in series. The overall effect of the $\beta(1, 1)$ priors on the final posterior system-reliability distribution is only slight, however, because of the strong prior data in the remaining subsystems. The $\beta(1, 1)$ priors yield a final $\beta(75.39, 94.38)$ system-reliability distribution, having .05, .5, and .95 quantiles of .38, .44, and .51, respectively. For the preceding reason, we recommend the use of Jeffreys's noninformative priors when few (if any) failures have been observed.

Because the choice of weights may be somewhat arbitrary, the sensitivity of the results to these choices is important. Table 3 gives the final posterior system-reliability distribution for several choices of weights, each, respectively, summing to 1. In Case 6, all of the weight is placed on the native system prior, thus w_{31} and w_{41} are irrelevant. The resulting posterior is not sensitive to the value of w_{31} when either $w_{41} = .25$ or $w_{41} = 0$ and $w_1 = .5$ (Cases 1, 2, and 3 and Cases 4, 12, and 13). The reason for this is the closeness of the induced and native priors as seen in Figure 5. The final posterior is only slightly sensitive to w_{41} when either $w_{31} = .5$ or $w_{31} = 0$ and $w_1 = .5$ (Cases 1, 4, and 5 and Cases 2, 12, and 14). The results are more sensitive to w_{41} when $w_1 = 1$ (Cases 7, 8, and 9) because all of the weight is given to the induced-system prior. Finally, as seen in Figure 8, the system posterior is quite sensitive to w_1 when $w_{31} = .5$ and $w_{41} = .25$ (Cases 1, 6, and 7), reflecting the difference in degree of belief embodied in the induced-system and native-system priors.

4. DISCUSSION

A Bayesian procedure for determining the reliability of a series system composed of binomial subsystems and components has been developed. The procedure uses both test and prior data at each of three levels in the system. Although the procedure is a Bayesian one based on subjective degree of belief (in the form of beta priors), an analysis using only binomial test data at the same three levels may also be performed. Noninformative component priors would be assigned, and zero weight would be given to the nonexistent native subsystem priors and native system prior in the analysis. The resulting posterior system distribution could then be used to provide reliability estimates that depend almost exclusively on the test results. Some degree of subjectivity still remains, however, as a result of the initial choice of noninformative component priors. Thus, in general, the final estimates will not necessarily agree with those produced by purely classical (non-Bayesian) methods.

To illustrate this, consider the missile subsystem in Table 1. Suppose now that we ignore the prior component data and use only the component and subsystem test results. The Lindstrom–Madden classical method (Lloyd and Lipow 1962) is perhaps the best-known approximate method for computing lower confidence limits on a series system of independent binomial components. Using this method, the missile component test data in Table 2 is equivalent to 22.77 successes in 25 tests. Thus, using only the component data, the maximum likelihood (ML) point estimate and 95% lower confidence bound on the missile reliability are .91 and .76, respectively. Combining this component system equivalent data with the actual missile test results yields an overall point estimate and 95% lower confidence bound on the missile reliability of .90 and .77, respectively.

Martz and Duran (1985) presented a Bayesian method based on Jeffreys's noninformative priors that produced lower probability bounds that were in good agreement with the Lindstrom–Madden lower confidence bounds. Consider, therefore, the Bayesian method presented here using $\beta(.5, .5)$ component priors. The p value for the induced missile subsystem prior approximate beta distribution in Step 3 is .99, indicating an excellent quality approximation. From Step 2 the induced-prior distribution on the missile reliability is a $\beta(28.85, 4.82)$ distribution, with a mean of .86 and a fifth percentile of .75. This lower Bayesian probability bound agrees quite well with the corresponding Lindstrom–Madden bound of .76. The point estimates differ primarily because of the optimistic nature of the ML estimator when no failures occur (as for components 1, 2, 5, and 6). These four components are effectively ignored in the ML point-estimate calculations, thus yielding an optimistic missile-reliability estimate. In contrast the Bayesian component point estimates are less optimistic for these four components, resulting in a less optimistic missile-reliability estimate. Including the missile test data in Step 5, the final posterior $\beta(35.85, 5.82)$ distribution on missile reliability has a mean of .86 and a fifth percentile of .76. Again this lower bound is close to the corresponding classical bound of .77.

Although only three levels have been considered here, the extension to more than three levels is straightforward: Steps 2–5 in Figure 2 are simply repeated for each additional level in the system. Thus a series system consisting of identifiable hardware or functional entities such as subcomponent, component, subsystem, system, supersystem, and so forth, for which either prior or test data have been obtained, can be analyzed using this procedure.

Finally, in all of the practical problems we have considered, the beta approximations in this procedure have been satisfactory in every case. Thompson and Haynes (1980) also reported success with this approximation in their examples.

ACKNOWLEDGMENTS

We are grateful to Robert Brownlee of Los Alamos National Laboratory for supplying the application on which the example in Section 3 is based and for his support of this work. We also thank an associate editor and two referees whose constructive comments greatly improved an earlier draft of this manuscript.

[Received September 1986. Revised June 1987.]

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