LA-UR-06-4520

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Intended for: Submitted to the International Journal of Approximate Reasoning, Special issue on the Imprecise Dirichlet Model

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Form 836 (7/06)

The Imprecise Dirichlet Model for Multilevel System Reliability

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Abstract

In this paper we expand on recent advances in the simultaneous inference for multilevel data in fault trees and Bayesian networks as discussed in [7] and [18]. Specifically, we incorporate partial knowledge of the parameters of a Dirichlet distribution. This is essentially a *credal network* with the information on the components represented by an Imprecise Dirichlet model. We make simultaneous inference on all nodes of the credal network by updating with data. We compare this model to existing Bayesian approaches with respect to the assumptions specific to the prior distribution and the available data.

Key words: Imprecise Dirichlet Model, Robust Bayesian Analysis, Fault Tree, Bayesian Network, Credal Network, Convex Sets of Probability Distributions, Reliability

1 Introduction

System reliability is defined as the ability of a system to perform a required function, under given environmental and operational conditions and for a stated period of time (cf. [11]). System reliability includes the reliability of all the subsystems and components that comprise the system under consideration. Several types of data can be used to model and analyze system reliability. These data may be obtained from different sources, such as observations

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from testing, computer simulations, or expert elicitation. There may be data on different levels of a system, including components, subsystems, or the full system. We refer to this last situation as *multilevel data*.

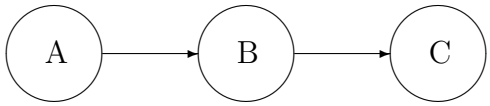
Multilevel data can occur in several ways. For example, we may have data on system tests but without component level information. Alternatively, we may have some prior component data from engineering tests that must be combined with full system data. Although this is a common occurrence for data in practice, inferential approaches for this kind of data are reasonably new [7, 18].

This article is organized as follows. Section 2 gives a brief background on imprecise probabilities (IPs) and the Imprecise Dirichlet model (IDM). Section 3 briefly reviews the use of Bayesian networks (BNs) for system representation. Sections 4, 5, and 6 present a sequence of examples structured to develop the use of the IDM with a BN for multilevel data for a system. Section 4 considers multilevel data on a fault tree modeled by a Bayesian multinomial-Dirichlet model. Section 5 expands the problem of Sect. 4 by allowing hyperpriors or second order probabilities. Section 6 unifies these methods under the IDM framework. Section 7 provides a discussion and comparison of the models.

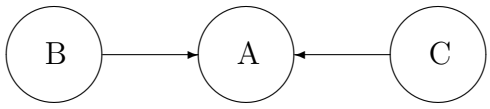
2 Imprecise Probabilities and the Imprecise Dirichlet Model

One of the attractions of IPs is the flexibility with which information can be represented. This uncertainty quantification method requires few assumptions on the structure of information and provides a very general characterization of uncertainty. The seminal work on IPs, Walley [14], is a comprehensive discussion on the foundation, interpretation, and application of IPs. The use of IPs for problems in reliability has been a subject of research in recent years. Interesting examples of IPs in reliability include [8], [13], and [4]. Our interest focuses on the use of data to update our current state of knowledge. Previous work in this area can be found in [14], [15], [16], and [5]. Similar work has been done with Dempster-Shafer belief structures in Wasserman [17]. Coolen [3] discusses the frequentist connection between empirical data and convex sets of probabilities.

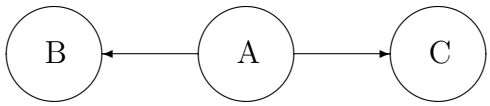
The IDM was first described by Walley [15]. Walley presents the case of making inferences about multinomial data with minimal prior information and compares the IDM with the objective Bayesian and frequentist models. Bernard [1] provides an excellent review of the IDM. When combined with Bayes' Theorem, the IDM allows for updating prior information with data.



(a) Serial: $\mathbf{P}(A, B, C) = \mathbf{P}(C|B)\mathbf{P}(B|A)\mathbf{P}(A)$



(b) Converging: $\mathbf{P}(A, B, C) = \mathbf{P}(A|B, C)\mathbf{P}(B)\mathbf{P}(C)$



(c) Diverging: $\mathbf{P}(A, B, C) = \mathbf{P}(C|A)\mathbf{P}(B|A)\mathbf{P}(A)$

Fig. 1. Specifying joint probability distributions using a Bayesian network.

3 Bayesian Networks for System Representation

Reliability block diagrams and fault trees are the most common representations in system reliability analysis. Their use is mainly to find the probability distribution for the state of the system. However, there are situations where these models do not offer enough flexibility to capture features of the system. BNs generalize fault trees and reliability block diagrams by allowing components and subsystems to be related by conditional probabilities instead of deterministic AND and OR relationships. Some recent work includes [12], [9], [10], and [18].

Formally, a BN is a pair $N = \langle (V, E), P \rangle$, where (V, E) are the nodes (vertices) and edges of a directed acyclic graph and P is a probability distribution on V. Each node contains a random variable, and the directed edges between them define conditional dependence or independence among the random variables. Figure 1 summarizes the three probabilistic relationships that can be specified in a BN. The key feature of a BN is that it specifies the joint distribution P over the set of nodes V in terms of conditional distributions. In particular, the joint distribution of V is given by

$$\prod_{v \in V} \mathbf{P}(v|\text{parents}[v]),$$

where the *parents* of a node are the set of nodes with an edge pointing to the node.

For example, in the serial structure in Fig. 1a, node A is called a *root* node.

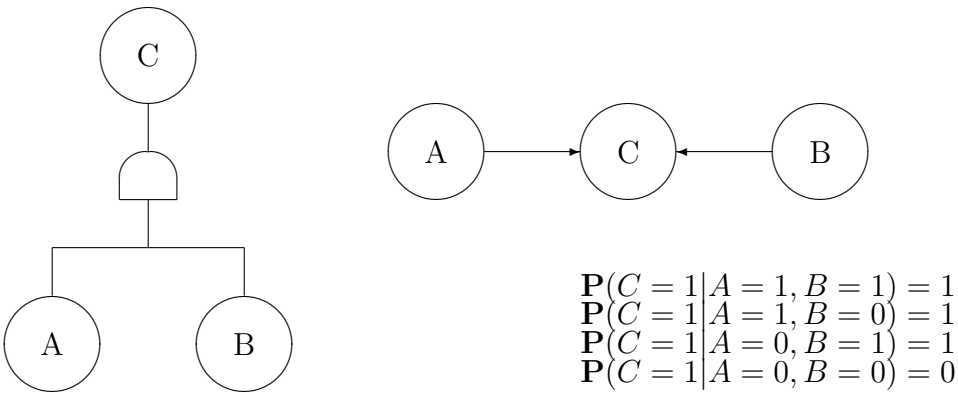


Fig. 2. Fault tree conversion to Bayesian network.

Node A is the parent (or predecessor) of node B, which is also called the *child* (or descendant) of node A. The directed branch linking the two nodes is interpreted to mean that the parent node has a direct influence on the child node. Node C has no descendants and it is called a *leaf*.

Figure 1b shows a converging BN while Fig. 1c shows a diverging BN. The primary applications of BNs have been to make inferences about node A given information about nodes B and C. Our concern is to combine data available on node A along with data available on nodes B and C to make inference about all three nodes.

BNs can be used as a direct generalization of fault trees. The fault-tree translation to a BN is straightforward, with the basic events that contribute to an intermediate event represented as parents and a child. Figure 2 shows the correspondence between a fault tree AND gate and a BN converging structure. Notice that a fault tree implies specific conditional probabilities. We can use the same BN converging structure of Fig. 2 with appropriate changes to the conditional probabilities to represent an OR gate.

One important innovation for the handling of multilevel data is found in the recent work on the Bayesian fault tree by Hamada et al. [7]. The authors develop a method for handling data that originates anywhere in a fault tree on basic events or intermediate events. While it is well known that basic events provide information on the higher level events, this work exploits the idea that higher level events also provide information on the basic events. This method uses Bayes' Theorem to obtain estimates of the basic event probabilities through the propagation of nonoverlapping data on the basic events and the top event. Wilson and Huzurbazar [18] generalize this method from the Bayesian fault tree to the BN. As a consequence, information can be propagated between levels in every direction of a more general modeling structure

Table 1
Multilevel Data

	R	Y	G
C_1	5	3	32
C_2	0	3	37
S	4	6	20

and simultaneous inference can be performed on all the nodes of the network. It is this versatility in the propagation of information that we apply to a BN with partial data represented as an imprecise probability distribution.

4 Bayesian Multinomial-Dirichlet Model for a Fault Tree with Multilevel Data

We illustrate statistical inference for multilevel data in system reliability with a multistate fault tree. Most previous applications have focused on dichotomous variables representing binary outcomes.

For this example, consider a fault tree representing a system, S, with two components, C_1 and C_2. The system and each component can be in one of three states: red or failed (R), yellow or degraded (Y), green or working (G). In a fault tree, if the states of the components are known, the state of the system is known. For this example, if any component is red, the system is red; if both components are green, the system is green; otherwise, the system is yellow.

Table 1 summarizes the data that was observed on the system. Forty tests ($\mathbf{X_1}$) were performed on C_1, 40 additional independent tests ($\mathbf{X_2}$) were performed on C_2, and 30 independent additional tests ($\mathbf{X_S}$) were performed on the system S. If the system was not green/working, no "failure attribution" was done, so there is no information about the states of C_1 and C_2 for the 30 tests.

We model the observed data using three multinomial distributions. Denote the sampling distribution for the data observed for C_1 as Multinomial(n_1, $(p_{1R}, p_{1Y}, 1-p_{1R}-p_{1Y})$). $P(C_1 = R)$ is denoted as p_{1R}, $P(C_1 = Y)$ is p_{1Y}, and $P(C_1 = G)$ is $p_{1G} = 1-p_{1R}-p_{1Y}$. Similarly, denote the sampling distribution for the data observed for C_2 as Multinomial(n_2, $(p_{2R}, p_{2Y}, 1-p_{2R}-p_{2Y})$) and for S as Multinomial(n_S, $(p_{SR}, p_{SY}, 1-p_{SR}-p_{SY})$).

Because the states of C_1 and C_2 determine the state of S, we can write

$$p_{SR} = p_{1R}p_{2R} + p_{1R}p_{2Y} + p_{1R}(1-p_{2R}-p_{2Y}) + p_{1Y}p_{2R} + \tag{1}$$

$$
\begin{aligned}
&(1-p_{1R}-p_{1Y})p_{2R}\\
p_{SY} &= p_{1Y}p_{2Y} + p_{1Y}(1-p_{2R}-p_{2Y}) + (1-p_{1R}-p_{1Y})p_{2Y}\\
p_{SG} &= (1-p_{1R}-p_{1Y})(1-p_{2R}-p_{2Y}).
\end{aligned}
$$

Consequently, the multinomial distribution describing the sampling distribution for the data observed for S has parameters: p_{1R}, p_{1Y}, p_{2R}, and p_{2Y}.

We are interested in using the data from Table 1 to make inferences about the reliability of C_1, C_2, and S, or, more generally, about the probability that the components and systems are in each of the three states. We use Bayesian methodology from [7] to perform these computations.

The sampling distribution of the data specifies the likelihood function,

$$
\begin{aligned}
L(p_{1R}, p_{1Y}, p_{2R}, p_{2Y} \,|\, \mathbf{x}) = {} & p_{1R}^5 p_{1Y}^3 (1-p_{1R}-p_{1Y})^{32}\\
& p_{2R}^0 p_{2Y}^3 (1-p_{2R}-p_{2Y})^{37} p_{SR}^4 p_{SY}^6 (1-p_{SR}-p_{SY})^{20},
\end{aligned}
$$

where expressions for p_{SR} and p_{SY} are given by (1).

We also specify prior distributions for the four unknown parameters. We use $(p_{1R}, p_{1Y}, p_{1G}) \sim \text{Dirichlet}(s_1 = 2, \mathbf{t_1} = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3}))$ and $(p_{2R}, p_{2Y}, p_{2G}) \sim \text{Dirichlet}(s_2 = 2, \mathbf{t_2} = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3}))$, using Walley's parameterization of the Dirichlet [16].

Using Bayes' Theorem, we know that the posterior distribution on the four unknown parameters, $f(p_{1R}, p_{1Y}, p_{2R}, p_{2Y} \,|\, \mathbf{x})$, is proportional to the product of the prior and the likelihood, so that

$$
\begin{aligned}
f(p_{1R}, p_{1Y}, p_{2R}, p_{2Y} \,|\, \mathbf{x}) \propto {} & p_{1R}^5 p_{1Y}^3 (1-p_{1R}-p_{1Y})^{32} p_{2Y}^3 (1-p_{2R}-p_{2Y})^{37}\\
& (p_{1R}p_{2R} + p_{1R}p_{2Y} + p_{1R}(1-p_{2R}-p_{2Y}) + p_{1Y}p_{2R} + (1-p_{1R}-p_{1Y})p_{2R})^4\\
& (p_{1Y}p_{2Y} + p_{1Y}(1-p_{2R}-p_{2Y}) + (1-p_{1R}-p_{1Y})p_{2Y})^6\\
& ((1-p_{1R}-p_{1Y})(1-p_{2R}-p_{2Y}))^{20}\\
& p_{1R}^{\frac{1}{3}} p_{1Y}^{\frac{1}{3}} (1-p_{1R}-p_{1Y})^{\frac{4}{3}} p_{2R}^{\frac{1}{3}} p_{2Y}^{\frac{1}{3}} (1-p_{2R}-p_{2Y})^{\frac{4}{3}}. \qquad (2)
\end{aligned}
$$

We want to calculate several quantities using (2). In particular, we want to calculate the marginal distribution for each parameter, which requires integrating out the other three parameters and normalizing the result. We would also like to calculate the marginal distributions for p_{SR}, p_{SY}, and p_{SG}, which are functions of the four parameters as given in (1). None of these distributions can be calculated easily by integration; even numerical integration techniques do not give us easy access to the marginal distributions for p_{SR}, p_{SY}, and p_{SG}.

One method that allows us to circumvent this difficulty is Markov chain

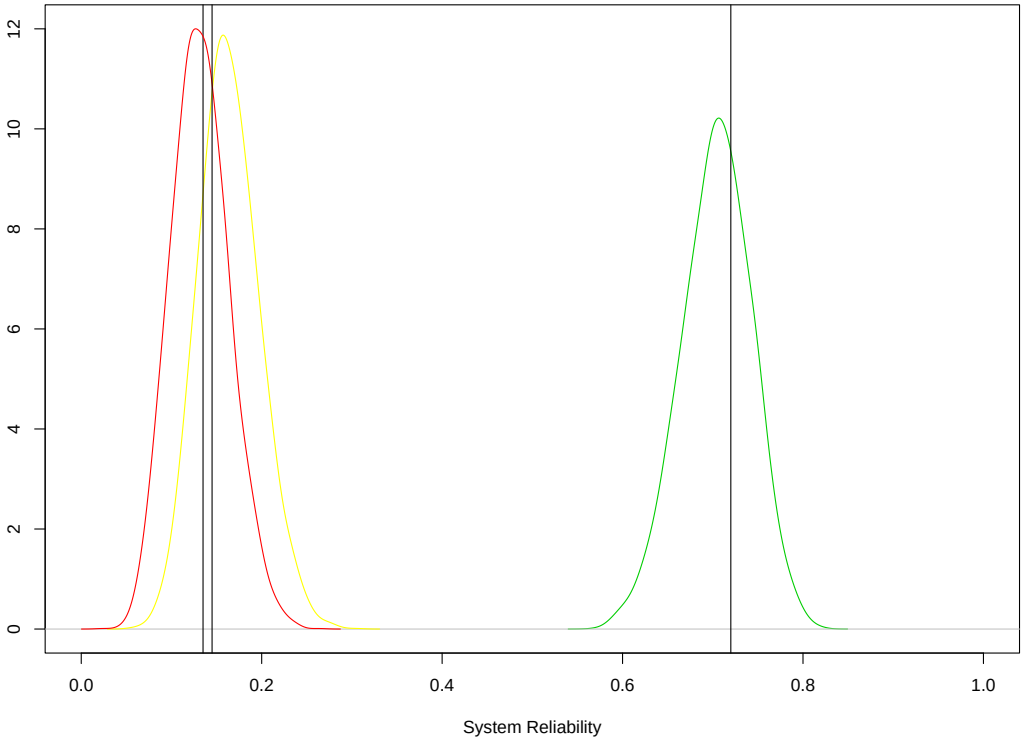


Fig. 3. The marginal posterior probability density functions for the multistate fault tree with specified prior distributions, with p_{SG} (green), p_{SY} (yellow), p_{SR} (red).

Monte Carlo (MCMC) [6]. MCMC constructs a stochastic process that has $f(p_{1R}, p_{1Y}, p_{2R}, p_{2Y} \,|\, \mathbf{x})$ as its limiting distribution. Once the algorithm has converged, it produces random samples from $f(p_{1R}, p_{1Y}, p_{2R}, p_{2Y} \,|\, \mathbf{x})$.

These random samples allow us to estimate the densities we want to calculate. Suppose that we generate m random samples from the MCMC algorithm. Denote the samples we generate $(p_{1R}^{(i)}, p_{1Y}^{(i)}, p_{2R}^{(i)}, p_{2Y}^{(i)})$, where $i = 1, \ldots, m$. We estimate the marginal distribution of p_{1R} with a histogram or kernel density estimator of $p_{1R}^{(i)}$, for $i = 1, \ldots, m$; we estimate the marginal distribution of p_{SG} with a histogram or kernel density estimator of $(1 - p_{1R}^{(i)} - p_{1Y}^{(i)})(1 - p_{2R}^{(i)} - p_{2Y}^{(i)})$, for $i = 1, \ldots, m$.

Figure 3 plots the posterior densities for p_{SG}, p_{SY}, and p_{SR}. Table 2 summarizes the posterior mean, 5th, and 95th percentiles for the system and component reliabilites.

In this example, we performed simultaneous inference on the reliability of the components and the system using data on the components and the system. In the fault-tree structure, if we know the state of the components, we know the state of the system. However, given the state of the system, we also have par-

Table 2
Summary Statistics for Multistate Fault Tree with Specified Prior Distribution

	Mean	5th Percentile	95th Percentile	Actual
p_{1R}	0.13	0.076	0.18	0.10
p_{1Y}	0.090	0.040	0.15	0.10
p_{1G}	0.78	0.72	0.84	0.80
p_{2R}	0.0061	0.0000020	0.027	0.05
p_{2Y}	0.095	0.051	0.15	0.05
p_{2G}	0.99	0.84	0.94	0.90
p_{SR}	0.13	0.082	0.19	0.145
p_{SY}	0.16	0.11	0.22	0.135
p_{SG}	0.70	0.64	0.77	0.72

tial knowledge about the states of the components. The approach here shows explicitly how we update our uncertainties about the component probabilities using both component and system data. (For another view of updating with incomplete observations, see [5]).

5 Bayesian Multinomial-Dirichlet Model with Hyperpriors for a Bayesian Network with Multilevel Data

As discussed in Sect. 3, BNs are a generalization of fault trees. In particular, the AND and OR relationships between components and subsystems and the full system are replaced with conditional probabilities. In this example, we consider a system represented as a converging BN (Fig. 1b) with the system as the child of components 1 (C_1) and 2 (C_2). Again, each component and the system can take on three states: red or failed (R), yellow or degraded (Y), green or working (G). Table 3 gives the conditional probabilities we use in this example, and Table 1 contains the data. We are interested in making inference about the probabilities that the components and system are in each of the three states.

We model the observed data using three multinomial distributions. Denote the sampling distribution for the data observed for C_1 as Multinomial(n_1, $(p_{1R}, p_{1Y}, 1 - p_{1R} - p_{1Y})$), the sampling distribution for the data observed for C_2 as Multinomial(n_2, $(p_{2R}, p_{2Y}, 1 - p_{2R} - p_{2Y})$), and the sampling distribution for S as Multinomial(n_S, $(p_{SR}, p_{SY}, 1 - p_{SR} - p_{SY})$).

The expressions for p_{SR} and p_{SY} are:

Table 3
Conditional Probabilities for Bayesian Network Examples

x	y	z	$P(S = x \mid C_1 = y, C_2 = z)$	x	y	z	$P(S = x \mid C_1 = y, C_2 = z)$
R	R	R	0.9	Y	Y	G	0.6
R	R	Y	0.7	Y	G	R	0.2
R	R	G	0.5	Y	G	Y	0.6
R	Y	R	0.6	Y	G	G	0.1
R	Y	Y	0.5	G	R	R	0.0
R	Y	G	0.2	G	R	Y	0.1
R	G	R	0.5	G	R	G	0.3
R	G	Y	0.2	G	Y	R	0.2
R	G	G	0.0	G	Y	Y	0.1
Y	R	R	0.1	G	Y	G	0.2
Y	R	Y	0.2	G	G	R	0.3
Y	R	G	0.2	G	G	Y	0.2
Y	Y	R	0.2	G	G	G	0.9
Y	Y	Y	0.4				

$$\begin{aligned} p_{SR} = & 0.9p_{1R}p_{2R} + 0.7p_{1R}p_{2Y} + 0.5p_{1R}p_{2G} + 0.6p_{1Y}p_{2R} + \\ & 0.5p_{1Y}p_{2Y} + 0.2p_{1Y}p_{2G} + 0.5p_{1G}p_{2R} + 0.2p_{1G}p_{2Y}, \end{aligned} \tag{3}$$

$$\begin{aligned} p_{SY} = & 0.1p_{1R}p_{2R} + 0.2p_{1R}p_{2Y} + 0.2p_{1R}p_{2G} + 0.2p_{1Y}p_{2R} + \\ & 0.4p_{1Y}p_{2Y} + 0.6p_{1Y}p_{2G} + 0.2p_{1G}p_{2R} + 0.6p_{1G}p_{2Y} \\ & 0.1p_{1G}p_{2G}. \end{aligned} \tag{4}$$

Using the data in Table 1, we write the likelihood function as

$$\begin{aligned} L(p_{1R}, p_{1Y}, p_{2R}, p_{2Y}) = & p_{1R}^5 p_{1Y}^3 (1 - p_{1R} - p_{1Y})^{32} p_{2Y}^3 (1 - p_{2R} - p_{2Y})^{37} \\ & p_{SR}^4 p_{SY}^6 (1 - p_{SR} - p_{SY})^{20}, \end{aligned} \tag{5}$$

where we substitute the expression (3) for p_{SR} and (4) for p_{SY}.

Next, we specify prior distributions for the four unknown parameters. We specify $(p_{1R}, p_{1Y}, p_{1G}) \sim \text{Dirichlet}(s_1 = 2, \mathbf{t_1})$ and $(p_{2R}, p_{2Y}, p_{2G}) \sim \text{Dirichlet}(s_2 = 2, \mathbf{t_2})$. In this example, $\mathbf{t_1}$ and $\mathbf{t_2}$ are unknown parameters, and we specify hyperpriors for them. In particular, we also use a Dirichlet distribution, with $\mathbf{t_1} \sim \text{Dirichlet}(2, (\frac{1}{6}, \frac{1}{6}, \frac{2}{3}))$ and $\mathbf{t_2} \sim \text{Dirichlet}(2, (\frac{1}{6}, \frac{1}{6}, \frac{2}{3}))$.

Using Bayes' Theorem, we know that the expression for the posterior distri-

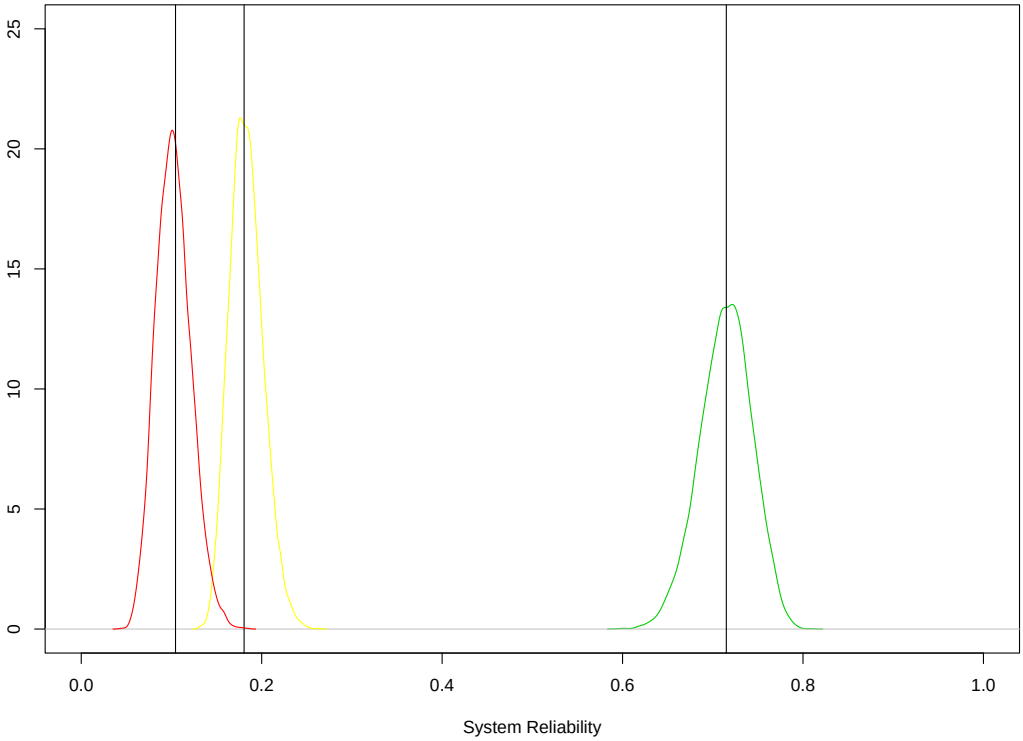


Fig. 4. The marginal posterior probability density functions for the BN with specified hyperprior, with p_{SG} (green), p_{SY} (yellow), p_{SR} (red).

bution is proportional to

$$\begin{aligned} f(p_{1R}, p_{1Y}, p_{2R}, p_{2Y}, \mathbf{t_1}, \mathbf{t_2} \,|\, \mathbf{x}) \propto\; & p_{1R}^5 p_{1Y}^3 (1 - p_{1R} - p_{1Y})^{32} \\ & p_{2Y}^3 (1 - p_{2R} - p_{2Y})^{37} p_{SR}^4 p_{SY}^6 (1 - p_{SR} - p_{SY})^{20} \\ & p_{1R}^{2t_{1R}} p_{1Y}^{2t_{1Y}} (1 - p_{1R} - p_{1Y})^{2t_{1G}} p_{2R}^{2t_{2R}} p_{2Y}^{2t_{2Y}} (1 - p_{2R} - p_{2Y})^{2t_{2G}} \\ & t_{1R}^{\frac{1}{3}} t_{1Y}^{\frac{1}{3}} t_{1G}^{\frac{4}{3}} t_{2R}^{\frac{1}{3}} t_{2Y}^{\frac{1}{3}} t_{2G}^{\frac{4}{3}}, \end{aligned} \tag{6}$$

where we substitute appropriate expressions for p_{SR} and p_{SY}.

Once again, the marginal densities are estimated using MCMC. Figure 4 plots posterior densities for the system, and Table 4 summarizes the posterior probabilites for all of the unknown parameters.

Table 4
Summary Statistics for BN with Specified Hyperprior

	Mean	5th Percentile	95th Percentile	Actual
p_{1R}	0.14	0.072	0.21	0.10
p_{1Y}	0.079	0.027	0.15	0.10
p_{1G}	0.78	0.71	0.85	0.80
p_{2R}	0.0043	0.000015	0.022	0.05
p_{2Y}	0.079	0.035	0.13	0.05
p_{2G}	0.92	0.86	0.96	0.90
p_{SR}	0.10	0.073	0.14	0.1045
p_{SY}	0.18	0.16	0.21	0.1805
p_{SG}	0.71	0.66	0.76	0.715
t_{1R}	0.083	0.000071	0.36	
t_{1Y}	0.038	0.0	0.21	
t_{1G}	0.88	0.43	1.0	
t_{2R}	0.027	0.000062	0.12	
t_{2Y}	0.062	0.000013	0.28	
t_{2G}	0.91	0.68	1.0	

6 Imprecise Dirichlet Model for a Bayesian Network with Multi-level Data

In this example, we generalize the example from Sect. 5 to use the IDM instead of a specific hyperprior. Again, we consider a converging BN with conditional probabilities given in Table 3, data in Table 1, and expressions for system reliability in (3). The observed data are modeled using three multinomial distributions, and we specify prior distributions $(p_{1R}, p_{1Y}, p_{1G}) \sim \text{Dirichlet}(2, \mathbf{t_1})$ and $(p_{2R}, p_{2Y}, p_{2G}) \sim \text{Dirichlet}(2, \mathbf{t_2})$.

In the previous example, we chose specific hyperpriors to describe our uncertainty about $\mathbf{t_1}$ and $\mathbf{t_2}$. In this example, we use the IDM. Instead of choosing a particular hyperprior, we allow $\mathbf{t_1}$ and $\mathbf{t_2}$ to take on any values such that $t_{1R} + t_{1Y} + t_{1G} = 1$ and $t_{2R} + t_{2Y} + t_{2G} = 1$. This specifies a *set* of prior distributions for $\mathbf{t_1}$ and $\mathbf{t_2}$ instead of a particular distribution. We apply Bayes' Theorem using each prior distribution from the set and the likelihood function specified in (5) to get a set of posterior distributions. We found the bounds by calculating the posterior distributions using MCMC for a fine grid of values for $\mathbf{t_1}$ and $\mathbf{t_2}$.

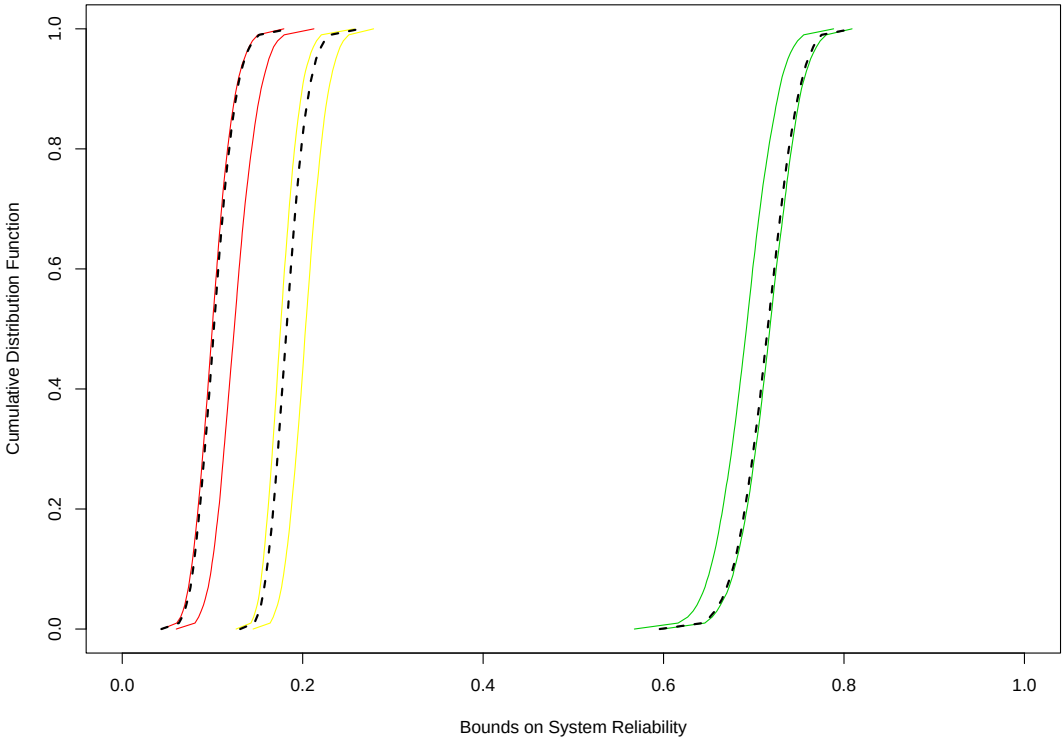


Fig. 5. The bounds on the cumulative posterior distribution functions for the BN with IDM, with p_{SG} (green), p_{SY} (yellow), p_{SR} (red). The posterior distributions from the BN with specified hyperprior are plotted as dashed lines.

We plot the bounds for the cumulative posterior distributions functions for p_{SR}, p_{SY}, and p_{SG} in Fig. 5. The posterior distributions from the previous example are plotted as dotted lines. Since these posterior distributions are a weighted combination of the posterior distributions from the IDM, they lie within the bounds. Table 5 summarizes the bounds on the posterior means for the set of posterior distributions.

7 Discussion

All of the examples considered in this paper consider simultaneous inference for multilevel data within a multistate system reliability problem. Independent data collected at both the components and the system are used to simultaneously update the probabilities that the components or system are in particular states. The example in Section 4 describes the system structure using a fault tree with known prior distributions for the states. The examples in Sections 5 and 6 use a BN to describe the system structure and consider a known hyperprior and the IDM to describe initial uncertainties about the parameters

Table 5
Summary Statistics for Posterior Means of BN with IDM

	Lower Bound Mean	Upper Bound Mean	Lower Bound 5th Quantile	Upper Bound 95th Quantile	Actual
p_{1R}	0.12	0.16	0.057	0.23	0.10
p_{1Y}	0.069	0.11	0.022	0.18	0.10
p_{1G}	0.76	0.79	0.69	0.86	0.80
p_{2R}	0.0027	0.041	0.000015	0.087	0.05
p_{2Y}	0.062	0.10	0.021	0.16	0.05
p_{2G}	0.89	0.92	0.83	0.96	0.90
p_{SR}	0.10	0.13	0.070	0.16	0.1045
p_{SY}	0.18	0.20	0.15	0.24	0.1805
p_{SG}	0.69	0.72	0.64	0.74	0.715

of the Dirichlet distributions describing the state probabilities.

Figure 4 plots the probability density functions for the posterior probabilities that the system is working, degraded, or failed. Since these posterior distribution are a weighted combination of the elements of the set of posterior distributions in Section 6, the cumulative distribution functions corresponding to the density functions in Figure 4 are plotted as dashed lines in Figure 5 and lie between the bounds on the set of posterior distributions.

Tables 4 and 5 summarize the results of analysis with a known hyperprior distribution and the IDM. The intervals in these tables have different interpretations. In Table 4, the intervals spanned by the 5th and 95th quantiles are a 90% posterior credible interval, which means that given our hyperprior and the data in Table 1, there is a 90% chance that the parameter lies within this interval. In Table 5, we summarize the minimum and maximum values taken on by the posterior means in the set of posterior distributions. In addition, we also provide a two-sided credible interval, as described in Walley [15]. These intervals are calculated as the smallest 5th quantile and the largest 95th quantile from the set of posterior distributions. These intervals will have at least 90% posterior probability for each distribution in the set. Notice that the interval of posterior means is too narrow and does not provide good coverage of the actual parameter values that were used to simulate the data in Table 1; the two-sided credible intervals are more reasonable, and are quite comparable to the results with a specified hyperprior.

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